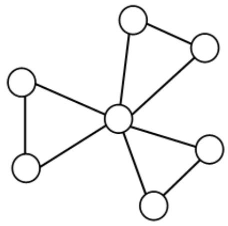
Solutions to question set for Chapter 4

1. Suppose a graph $G = (V, E)$, with $|V| \geq 3$, has the following properties:

   (a) For any two vertices x and y there is exactly one other vertex z which is adjacent to both x and y;

   (b) There is some vertex v which is adjacent to all others.

   Prove that G must consist of a collection of triangles joined at a common 'hub' vertex, as in the example on the right.
   (The famous **Friendship Theorem** of Paul Erdős, Alfréd Rényi and Vera Sós says that, in fact, property (a) above implies property (b)! See more here: Theorem of the Day)

SOLUTION: Let x be any vertex other than the hub vertex v. Now x and v must share exactly one common 'neighbour' say, y, to which they are both adjacent. Since v is adjacent to both x and y by (2) it is enough for there to be an edge joining x and y. Thus x, y and v form a triangle. Moreover, neither x nor y can be adjacent to any other vertex. For suppose x, say, was adjacent to $z \neq y, v$. Then z is adjacent to v by (2), but then v and x share *two* common neighbours, in contradiction of (1). Hence the vertex set must be partitioned into triangles sharing a common vertex v.

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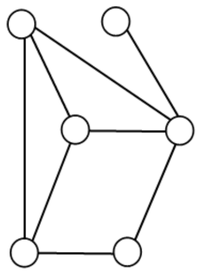
2. A non-increasing sequence of non-negative integers is a **degree sequence** if some graph G has these integers as its vertex degrees. For example, the sequence $(4, 3, 3, 3, 2, 1)$ is a degree sequence because these are precisely the vertex degrees of the graph shown on the right.

   A 1960 theorem of Paul Erdős and Tibor Gallai says that a non-increasing sequence of non-negative integers $(d_1, d_2, \ldots, d_n)$ is a degree sequence if and only if $\sum d_i$ is even and, for all k, $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i - \sum_{i=k+1}^{n} \min(d_i, k) \leq k(k-1). \quad (1)$$

   (a) Why do we require that $\sum d_i$ is even?

   (b) Confirm that the degree sequence $(4, 3, 3, 3, 2, 1)$ satisfies inequality (1).

   (c) What is the complexity of testing this inequality for a sequence of length n?

SOLUTION:

(a) $\sum d_i$ is even because the Handshaking Lemma says this is the case.

(b) The inequalities are as follows:

k

$1:\ 4 - \big(\min(3,1) + \min(3,1) + \min(3,1) + \min(2,1) + \min(1,1)\big) = 4 - (1+1+1+1+1) = -1 \leq 1 \times 0 = 0$

$2:\ 4 + 3 - \big(\min(3,2) + \min(3,2) + \min(2,2) + \min(1,2)\big) = 7 - (2+2+2+1) = 0 \leq 2 \times 1 = 2$

$3:\ 4 + 3 + 3 - \big(\min(3,3) + \min(2,3) + \min(1,3)\big) = 10 - (3+2+1) = 4 \leq 3 \times 2 = 6$

$4:\ 4 + 3 + 3 + 3 - \big(\min(2,4) + \min(1,4)\big) = 13 - (2+1) = 10 \leq 4 \times 3 = 12$

$5:\ 4 + 3 + 3 + 3 + 2 - \min(1,5) = 15 - 1 = 14 \leq 5 \times 4 = 20$

$6:\ 4 + 3 + 3 + 3 + 2 + 1 = 16 \leq 6 \times 5 = 30$

(c) We are investigating the complexity of an arithmetic calculation, so we do not count the 'number of steps' but instead the number of additions, min calculations and comparisons ($\leq$).

For inequality k we have n additions (from $\sum_{i=1}^{k}$ and $\sum_{i=k+1}^{n}$), $n-k$ mins and 1 comparison. So the total over $k = 1, \ldots, n$ is n^2 additions, $n-1+n-2+n-3+\ldots+2+1$ mins and n comparisons.

This gives a total of $n^2 + \frac{1}{2}n(n-1) + n$ arithmetic operations, which is $O(n^2)$.

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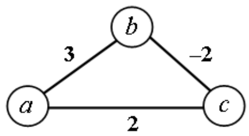
3. Dijkstra's algorithm fails to work properly if negative edge-weights are allowed. Consider the following strategy which attempts to overcome this problem:

    Let ω be the negative weight of greatest absolute value. Add $|\omega| + 1$ to every weight to create a graph with only positive edge weights. Run Dijkstra's algorithm on the result. Now subtract $|\omega| + 1$ from every edge of the resulting tree.

    Does this strategy work?

SOLUTION: No! Look at the example on page 1 of Week 4, Lecture notes 1:



The strategy dictates that we add 3 to every edge. But this means longer paths will have more multiples of 3 added to their total weight, cancelling out the effect of negative edges. The path ab, bc in the graph has weight $3-2 = 1$ which is less than ac with weight 2. But adding 3 to all edges makes the comparison $(3+3)+(-2+3) = 1+6$ against $2+3 = 5$. Again the path ac is chosen but this is not the shortest path in the original graph.

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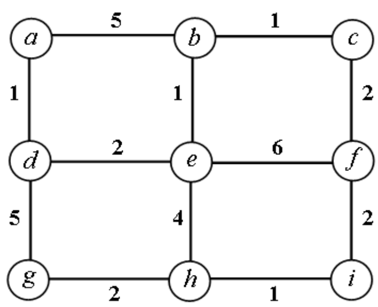
4. **[FOR MARKING]**

    (a) Run Dijkstra's algorithm on the following graph G using the root vertex $v = a$. Show, diagrammatically or by means of a table, how the algorithm operates at each iteration.

    Give the final tree and table of distances which are the outputs of the algorithm.

    (b) Draw a minimum-weight spanning tree for the graph G in part (a). You need not show how you obtained this tree.

    (c) Is your tree in part (a) a minimum-weight spanning tree for G? Does your tree in part (b) give minimum-weight paths from a to all other vertices? Give counterexamples where appropriate.



A jpeg version of the graph for this question, which you may care to reproduce in your answer, is available on the Week 4 page of the MTH6105 website.

SOLUTION:

(a) The diagrammatic version is:

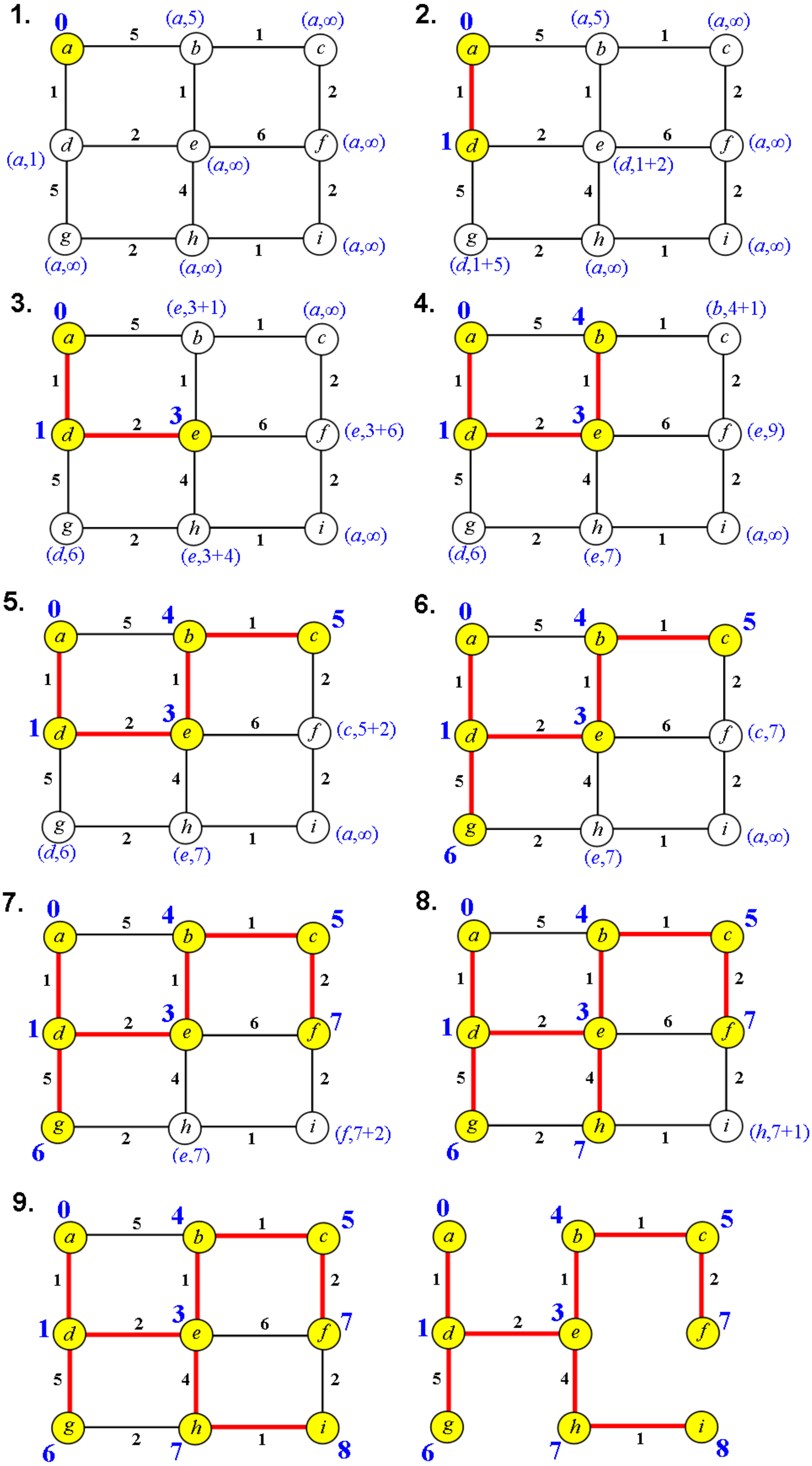
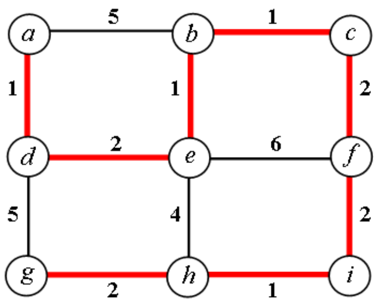


Table of distances: x	a	b	c	d	e	f	g	h	i
$\mathrm{d}(a,x)$	0	4	5	1	3	7	6	7	8

The table version of the solutions is:

	a	b	c	d	e	f	g	h	i	u	tree edges
	0	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	a	$\emptyset$
1		$(a,\ 5)$	$(a,\ \infty)$	$(a,\ 1)$ **1**	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	$(a,\ \infty)$	d	$\{ad\}$
2		$(a,\ 5)$	$(a,\ \infty)$		$(d,\ 1+2)$ **3**	$(a,\ \infty)$	$(d,\ 1+5)$	$(a,\ \infty)$	$(a,\ \infty)$	e	$\{ad,\ de\}$
3		$(e,\ 3+1)$ **4**	$(a,\ \infty)$			$(e,\ 3+6)$	$(d,\ 6)$	$(e,\ 3+4)$	$(a,\ \infty)$	b	$\{ad,\ de,\ be\}$
4			$(b,\ 4+1)$ **5**			$(e,\ 9)$	$(d,\ 6)$	$(e,\ 7)$	$(a,\ \infty)$	c	$\{ad,\ de,\ be,$ $bc\}$
5						$(c,\ 5+2)$	$(d,\ 6)$ **6**	$(e,\ 7)$	$(a,\ \infty)$	g	$\{ad,\ de,\ be,$ $bc,\ dg\}$
6						$(c,\ 7)$ **7**		$(e,\ 7)$	$(a,\ \infty)$	f	$\{ad,\ de,\ be,$ $bc,\ dg,\ cf\}$
7								$(e,\ 7)$ **7**	$(f,\ 7+2)$	h	$\{ad,\ de,\ be,$ $bc,\ dg,\ cf$ $eh\}$
8									$(h,\ 7+1)$ **8**	i	$\{ad,\ de,\ be,$ $bc,\ dg,\ cf$ $eh, hi\}$

(b) A minimum-weight spanning tree is as shown below:



(c) The solution tree in part (a) is not a minimum-weight spanning tree since it has total weight 17 whereas the tree in part (b) has total weight 12.

The minimum-weight spanning tree does not give minimum distances. For example, the distance from a to i in this tree is 9, whereas the tree in part (a) reveals that $d_G(a, i) = 8$.