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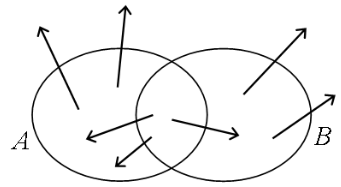
Question set for Chapter 5

1. Let G be a digraph in which $d^-(x) = d^+(x)$ for all vertices $x \in V(G)$. Prove that, for any subset $U \subseteq V(G)$, the number of edges directed out of U equals the number of edges directed into U.

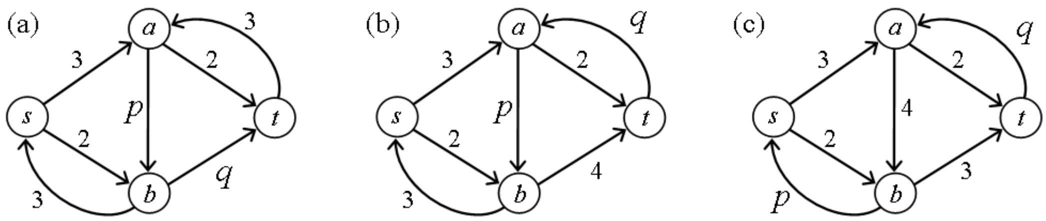
   [HINT: apply the net flow lemma which was proved in Chapter 5, Section 2.]

2. In contrast to the well-known 'mod function' identity concerning the cardinality of finite sets:
$$|A \cup B| = |A| + |B| - |A \cap B|,$$
   a function f defined on sets is called **submodular** if, for any sets A, B,
$$f(A \cup B) \leq f(A) + f(B) - f(A \cap B). \qquad (1)$$
   Define the function $d^+(X)$ on subsets X of vertices in a digraph in the natural way: $d^+(X)$ is the number of edges leaving X. It may be shown that this function satisfies inequality (1).

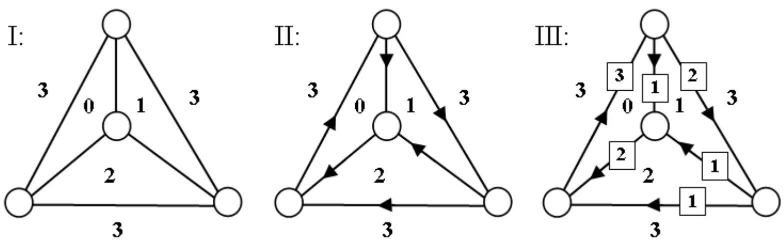
   In the picture on the right the function $d^+(X)$ is shown schematically for two vertex subsets A and B in a digraph. Write down
$$d^+(A), \qquad d^+(B), \qquad d^+(A \cap B), \qquad d^+(A \cup B).$$
   Hence confirm that (1) holds with equality. What is missing from the picture which explains why the inequality (1) may be strict?

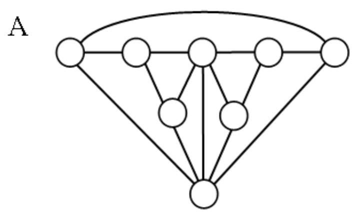


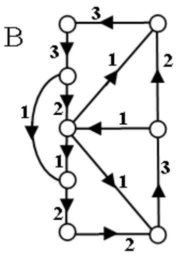
3. In each of the diagrams below, specify the values which p and q must necessarily have, in order for the edge labels to represent a valid st-flow.



4. We have no new algorithm to run yet, so this is a bit of general graph theory... Let G be a simple graph which is drawn in the plane with no crossing edges. Any cycle enclosing a region of the plane containing no other vertices is called a **face**. Now do the following: **Step I:** Number the faces of G using $0, \ldots, k-1$, for some k, in such a way that no edge belongs to two faces numbered the same. See the example below. Note the 'outside' face numbered 3 (the '3' is repeated several times but it is all one face).







   Step II: Put an arrow on each edge so that if you follow the arrow the larger face number lies on your left.**Step III**: Number each edge with the difference of its two face numbers. Amazingly, you have a network flow! (The s and t don't matter—all vertices have net flow zero).

   (a) Repeat Steps I–III for the graph at A.

   (b) Reverse the Steps I–III for the graph at B: give the outside face the number zero and use the flow to determine the other face numbers, one by one.

   (c) Prove that Steps I–III always give a flow. This is trickier, but an informal proof will do. It is part of some theory which says, among other things, that the Four-Colour Theorem (see Theorem of the Day) is equivalent to saying every bridgeless planar graph can be given edge directions and a flow using the edge values 1, 2 and 3.