Solutions to question set for Chapter 5

1. Let G be a digraph in which $d^-(x) = d^+(x)$ for all vertices $x \in V(G)$. Prove that, for any subset $U \subseteq V(G)$, the number of edges directed out of U equals the number of edges directed into U.

   [HINT: apply the net flow lemma which was proved in Chapter 5, Section 2.]

   SOLUTION: Consider G as a network. Give each edge e a flow value $f(e) = 1$. Then $d^-(x) = d^+(x) \Rightarrow f^-(x) = f^+(x)$ for all vertices x. So this is a valid flow function. And we have

   $$\text{for } X \subseteq V(G),\ f^+(U) = \text{ no. of edges leaving } X, \text{and } f^-(U) = \text{ no. of edges entering } X. \qquad (*)$$

   Now let $U \subseteq V(G)$ and let s be any vertex in U. By the Lemma, we have

   $$f^+(U) - f^-(U) = f^+(s) - f^-(s).$$

   But $f^+(s) - f^-(s) = d^+(s) - d^-(s) = 0$, so $f^+(U) - f^-(U) = 0$, and the result follows by $(*)$.

2. In contrast to the well-known 'mod function' identity concerning the cardinality of finite sets:

   $$|A \cup B| = |A| + |B| - |A \cap B|,$$

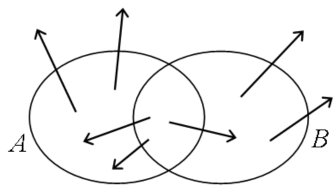
   a function f defined on sets is called **submodular** if, for any sets A, B,

   $$f(A \cup B) \le f(A) + f(B) - f(A \cap B). \qquad (1)$$

   Define the function $d^+(X)$ on subsets X of vertices in a digraph in the natural way: $d^+(X)$ is the number of edges leaving X. It may be shown that this function satisfies inequality (1).

   In the picture on the right the function $d^+(X)$ is shown schematically for two vertex subsets A and B in a digraph. Write down

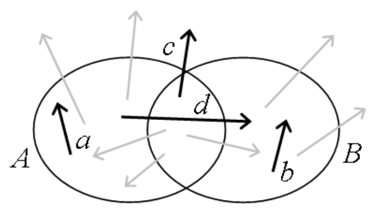
   $d^+(A)$, $d^+(B)$, $d^+(A \cap B)$, $d^+(A \cup B)$.

   Hence confirm that (1) holds with equality. What is missing from the picture which explains why the inequality (1) may be strict?

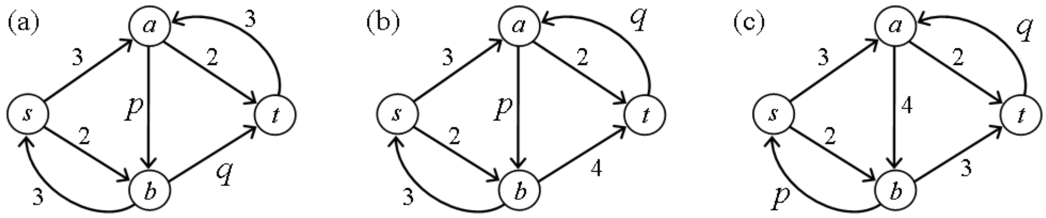
   SOLUTION:
   The edges as shown give equality: $d^+(A) = 3$, $d^+(B) = 4$, $d^+(A \cup B) = 4$, $d^+(A \cap B) = 3$. So

   $$d^+(A \cup B) = d^+(A) + d^+(B) - d^+(A \cap B).$$

   Various possible edges can be added to the diagram as shown on the right. Of these, a and b do not contribute to d^+ of any set in the inequality; edge c adds 1 to both sides of the inequality and so does not change it. But edge d increases the RHS of the inequality and not the LHS, so this gives strict inequality.

3. In each of the diagrams below, specify the values which p and q must necessarily have, in order for the edge labels to represent a valid st-flow.



SOLUTION:

(a) $p = 4$ to balance net flow at a:

$$f^{-}(a) = 3 + 3 = 6,\ \ f^{+}(a) = p + 2 = 4 + 2 = 6.$$

Then $q = 3$ to balance net flow at b:

$$f^{-}(b) = 2 + p = 2 + 4 = 6,\ \ f^{+}(b) = 3 + q = 3 + 3 = 6.$$

(b) $p = 5$ to balance net flow at b:

$$f^{-}(b) = 2 + p = w + 5 = 7,\ \ f^{+}(b) = 3 + 4 = 7.$$

Then $q = 4$ to balance net flow at a:

$$f^{-}(a) = q + 3 = 4 + 3 = 7,\ \ f^{+}(a) = p + 2 = 5 + 2 = 7.$$

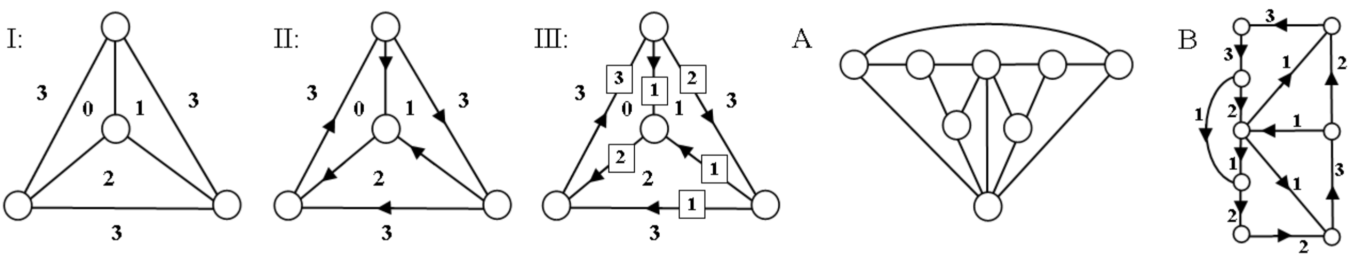
(c) $p = 3$ to balance net flow at b:

$$f^{-}(b) = 2 + 4 = 6,\ \ f^{+}(b) = 3 + p = 3 + 3 = 6.$$

And $q = 3$ to balance net flow at a:

$$f^{-}(a) = 3 + q = 3 + 3 = 6,\ \ f^{+}(a) = 4 + 2 = 6.$$

4. We have no new algorithm to run yet, so this is a bit of general graph theory... Let G be a simple graph which is drawn in the plane with no crossing edges. Any cycle enclosing a region of the plane containing no other vertices is called a **face**. Now do the following: **Step I:** Number the faces of G using $0, \ldots, k-1$, for some k, in such a way that no edge belongs to two faces numbered the same. See the example below. Note the 'outside' face numbered 3 (the '3' is repeated several times but it is all one face).



Step II: Put an arrow on each edge so that if you follow the arrow the larger face number lies on your left. And **Step III**: Number each edge with the difference of its two face numbers. Amazingly, you have a network flow! (The s and t don't matter—all vertices have net flow zero).

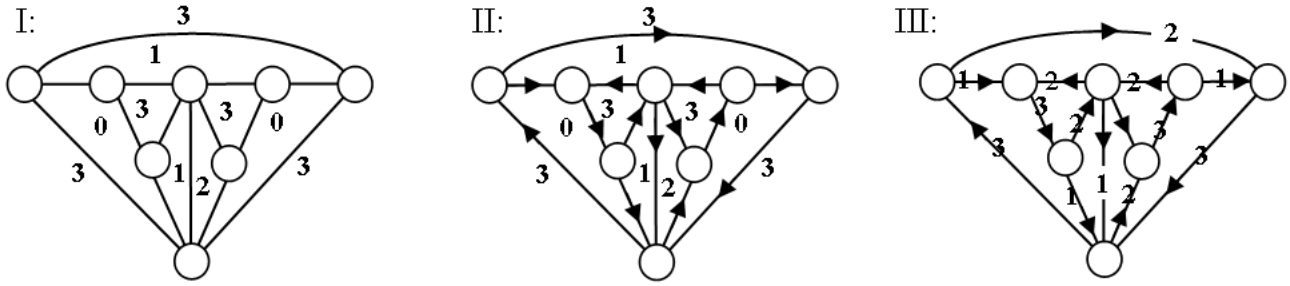
(a) Repeat Steps I–III for the graph at A.

(b) Reverse the Steps I–III for the graph at B: give the outside face the number zero and use the flow to determine the other face numbers, one by one.

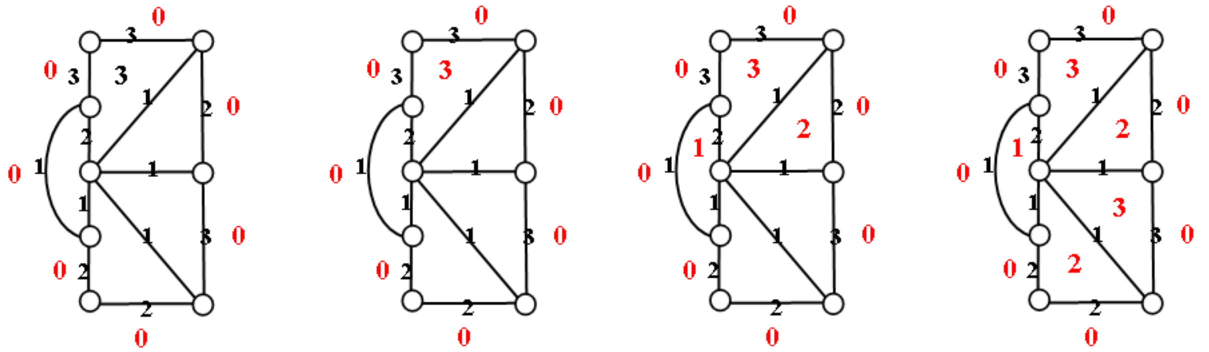
(c) Prove that Steps I–III always give a flow. This is trickier, but an informal proof will do. It is part of some theory which says, among other things, that the Four-Colour Theorem (see Theorem of the Day) is equivalent to saying every bridgeless planar graph can be given edge directions and a flow using the edge values 1, 2 and 3.

SOLUTION:

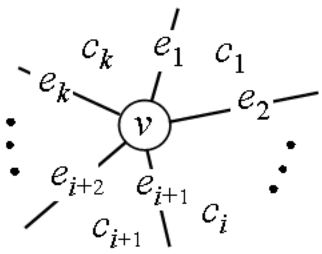
(a) We can find a numbering using $0, \ldots, 3$ as shown in I below. Steps II and III are then fixed:



(b) In the images below we have numbered the outside face 0 and then numbered 1 inside face $3 - 0 = 3$. This numbers two adjacent faces and thereafter the final two faces are fixed.



(c) We must show that the net flow $f^+(v) - f^{(}v)$ at any vertex v is zero. Suppose $d(v) = k$. Then k faces of the graph meet at v and these are numbered, say, $c_1, c_2, \ldots, c_k$, as shown right, with the condition that $c_{i-1} \neq c_i \neq c_{i+1}$, with the subscripts taken $\bmod k$ (so c_k is adjacent to c_{k-1} and $c_{k+1} = c_1$).



Now we can calculate

$$f^+(v) - f^-(v) = a_1b_1(c_1 - c_2) + a_2b_2(c_2 - c_3) + \ldots + a_kb_k(c_k - c_1),$$

where $a_1 = \pm 1$ according as $c_i > c_{i+1}$ or not; and $b_i = \pm 1$ according as e_{i+1} is directed out of or into v. But now we observe that $c_i < c_{i+1}$ if and only if e_{i+1} is directed into v, so that $a_ib_i = 1$ for all i. So

$$f^+(v) - f^-(v) = c_1 - c_2 + c_2 - c_3 + \ldots + c_k - c_1 = 0,$$

as required.

(Definitely above the level of this book, but I hope that the reason it works was discoverable even if the formal explanation might not have been!)