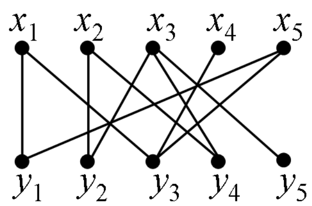
Question set for Chapter 8

1. Recall that, for two subsets X and Y of a set A, the **symmetric difference** $X\Delta Y$ is defined to be $(X\backslash Y)\cup(Y\backslash X)$: the subset of those elements of A which are in X or Y but not both.

   Let G be any graph and let M be a matching in G. Suppose that G has an M-augmenting path P. Prove that $M' = M\Delta P$ is a matching of cardinality $|M'| = |M| + 1$.

2. A theorem of Phillip Hall says that a bipartite graph G with parts X and Y has a matching which saturates every vertex of X if and only if, for every subset S of X, we have $|N(S)| \geq |S|$, where $N(S)$ denotes the set of all vertices of G adjacent to vertices in S.

   Find a subset S of the vertices $\{x_1, \ldots, x_5\}$ in the graph on the right which fails to meet this condition. Why does this automatically mean that some subset of $\{y_1, \ldots, y_5\}$ must also fail the condition?

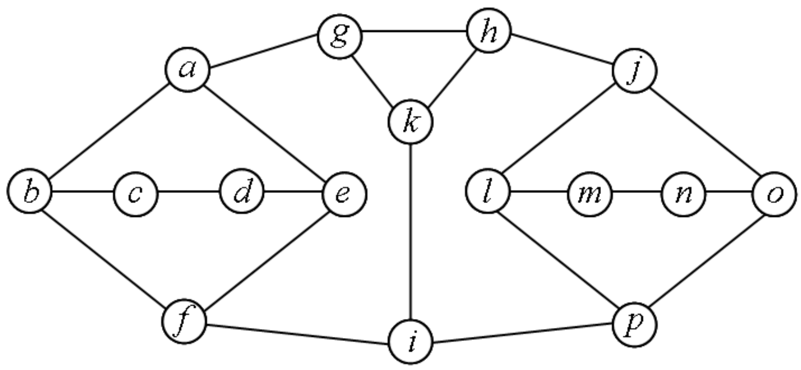
3. Consider the 3×5 matrix below:

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

   We can view this matrix as encoding a bipartite graph: the parts of the bipartite graph are $\{x_1, \ldots, x_3\}$ and $\{y_1, \ldots, y_5\}$, and there is an edge joining x_i and y_j if and only if there is a 1 in row i, column j.

   (a) Draw the bipartite graph G corresponding to this matrix.
   (b) Give the 5×5 matrix which encodes the graph in Question 2.
   (c) What subsets of matrix entries correspond to matchings in G?

4. Again, we do not yet have an algorithm to practice, so we will investigate the proof that absence of an augmenting path implies maximality. We shall work with the following graph and with two matchings M and M':



$$M = \{bc, ef, hj, ik, lp, no\}$$
$$M' = \{ae, bf, cd, hk, ip, jl, no\}$$

   (a) Say giving brief reasons whether each of M and M' is
   (i) maximal; (ii) maximum; (iii) perfect.

   (b) The proof that there must be an M-augmenting path because $|M'| > |M|$ is constructive:
   (i) Draw a new graph H which consists of the edges in the symmetric difference
   $$M'\Delta M = M'\backslash M \cup M\backslash M'.$$
   (ii) Label the vertices of your graph H with their degrees in H to confirm that all the degrees are 1 or 2 (and therefore form a collection of paths and cycles).
   (iii) Draw the cycles and paths you find in your graph H, clearly labelling their vertices, and find a path which begins and ends with an M' edge.
   (iv) Identify this path in the original graph as given above and specify the matching $P\Delta M$ which is necessarily bigger than M.

   (c) Give an augmenting path for the new matching you found in part (b) and use it supply a maximum matching for G. Also find a minimum cover for G and comment on the relationship between $\nu(G)$ and $\tau(G)$ for the given graph G.