Solutions to question set for Chapter 8

1. Recall that, for two subsets X and Y of a set A, the **symmetric difference** $X\Delta Y$ is defined to be $(X\backslash Y) \cup (Y\backslash X)$: the subset of those elements of A which are in X or Y but not both.

   Let G be any graph and let M be a matching in G. Suppose that G has an M-augmenting path P. Prove that $M' = M\Delta P$ is a matching of cardinality $|M'| = |M| + 1$.

SOLUTION: We must show (1) that $M' = M\Delta P$ is a matching, and (2) that $|M'| = |M| + 1$.

(1): M' is a matching if (i) it is a subset of $E(G)$ and (ii) no vertex of G belongs to more than one edge of M'.

(i): Since $M' = M\Delta P = (M\backslash P) \cup (P\backslash M)$ which is a union of two subsets of $E(G)$, it follows that M' is a subset of $E(G)$.

(ii): suppose some vertex belongs to two edges, say e_1 and e_2 of M'. Since $M\backslash P$ and $P\backslash M$ are disjoint, either (I) both edges are in $M\backslash P$ or (II) both are in $P\backslash M$ or (III) one edge is in each set, say, $e_1 \in M\backslash P$ and $e_2 \in P\backslash M$. Now (I) is impossible because M is a matching; (II) is impossible because P is alternating, so that every vertex of P except the first and last belong to an edge of M, so no vertex can have degree 2 in $P\backslash M$. And (III) implies that some vertex of P is adjacent to an edge of M not lying in P. Since P is alternating this can only be the first or last vertex but these vertices are M-unsaturated by definition. So (III) is also impossible. So no vertex belongs to two edges of M'.

(2): An M-augmenting path has one more non-M edge than it has M-edges. So $P\backslash M$ has cardinality one greater than the number of M-edges in P. All edges of M not in P are retained in M'. So M' has one more edges than M.

[This was part (c) of a question on the 2010 exam paper and was worth 10 marks, 6 marks for (1) and 4 for (2). The argument presented above is quite formal — the majority of the marks were awarded to anyone who realised that a two-part argument was required and gave any reasonably logical version of this argument.]

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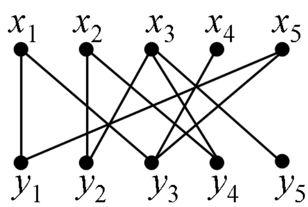
2. A theorem of Phillip Hall says that a bipartite graph G with parts X and Y has a matching which saturates every vertex of X if and only if, for every subset S of X, we have $|N(S)| \geq |S|$, where $N(S)$ denotes the set of all vertices of G adjacent to vertices in S.

   Find a subset S of the vertices $\{x_1, \ldots, x_5\}$ in the graph on the right which fails to meet this condition. Why does this automatically mean that some subset of $\{y_1, \ldots, y_5\}$ must also fail the condition?



SOLUTION: The set $S = \{x_1, x_4, x_5\}$ has $N(S) = \{y_1, y_3\}$. Since $|N(S)| = 2 < |S|$, the set S fails the condition of Hall's theorem.

Let $X = \{x_1, \ldots, x_5\}$ and $Y = \{y_1, \ldots, y_5\}$. If no subset of Y fails the condition of Hall's theorem then there is a matching M which saturates every vertex of Y. Then $|M| = 5$; every edge of M includes a vertex of X and no vertex of X can lie in two edges of M. Therefore M saturates every vertex in X which we know is impossible. So some subset of Y fails the condition of Hall's theorem. (Such a set is $\{y_2, y_4, y_5\}$).

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3. Consider the 3×5 matrix below:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We can view this matrix as encoding a bipartite graph: the parts of the bipartite graph are $\{x_1, \ldots, x_3\}$ and $\{y_1, \ldots, y_5\}$, and there is an edge joining x_i and y_j if and only if there is a 1 in row i, column j.

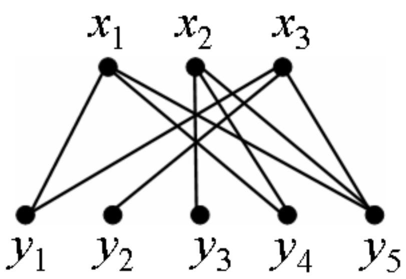
(a) Draw the bipartite graph G corresponding to this matrix.

(b) Give the 5×5 matrix which encodes the graph in Question 2.

(c) What subsets of matrix entries correspond to matchings in G?

SOLUTION:

(a) The bipartite graph corresponding to the matrix is:



(b) The matrix corresponding to the bipartite graph is:

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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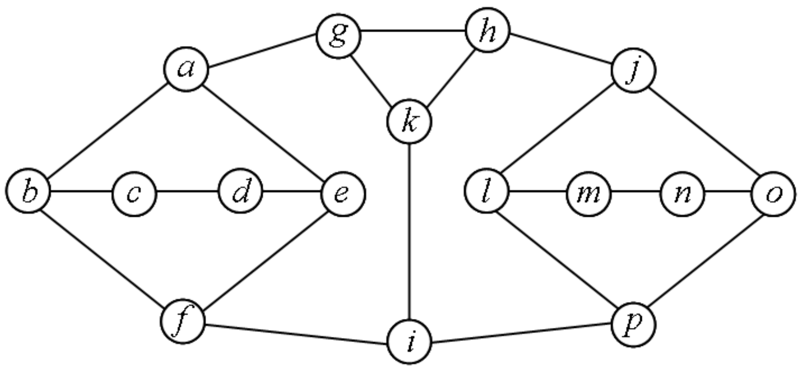
4. Again, we do not yet have an algorithm to practice, so we will investigate the proof that absence of an augmenting path implies maximality. We shall work with the following graph and with two matchings M and M':



$$M = \{bc,\ ef,\ hj,\ ik,\ lp,\ no\}$$
$$M' = \{ae,\ bf,\ cd,\ hk,\ ip,\ jl,\ no\}$$

(a) Say giving brief reasons whether each of M and M' is

(i) maximal; (ii) maximum; (iii) perfect.

(b) The proof that there must be an M-augmenting path because $|M'| > |M|$ is constructive:

(i) Draw a new graph H which consists of the edges in the symmetric difference
$$M' \Delta M = M' \backslash M \cup M \backslash M'.$$

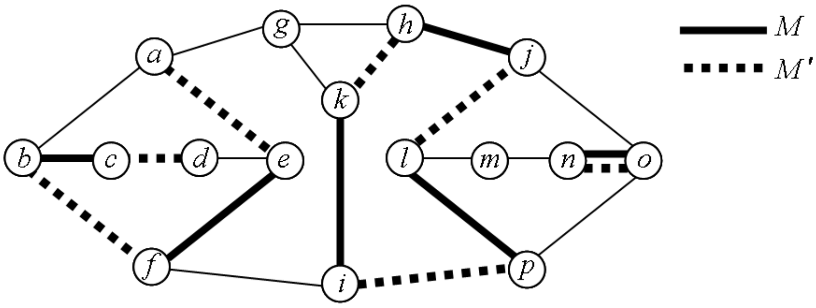
(ii) Label the vertices of your graph H with their degrees in H to confirm that all the degrees are 1 or 2 (and therefore form a collection of paths and cycles).

(iii) Draw the cycles and paths you find in your graph H, clearly labelling their vertices, and find a path which begins and ends with an M' edge.

(iv) Identify this path in the original graph as given above and specify the matching $P \Delta M$ which is necessarily bigger than M.

(c) Give an augmenting path for the new matching you found in part (b) and use it supply a maximum matching for G. Also find a minimum cover for G and comment on the relationship between $\nu(G)$ and $\tau(G)$ for the given graph G.

SOLUTION: (a) Something like the following drawing maybe helpful (but is not required by the questio



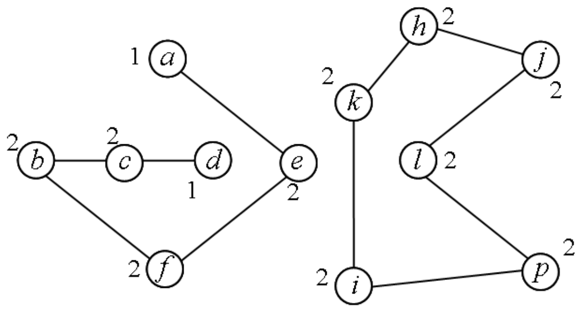
(i) M is not maximal since the edge ag shares neither endpoint with any edge in M; M' is maximal since every edge of G has at least one endpoint in M'.

(ii) M is not maximum since M' is larger than M. Also M' is not maximum since G contains a perfect matching (which will be constructed in part (c) but is not hard to spot).

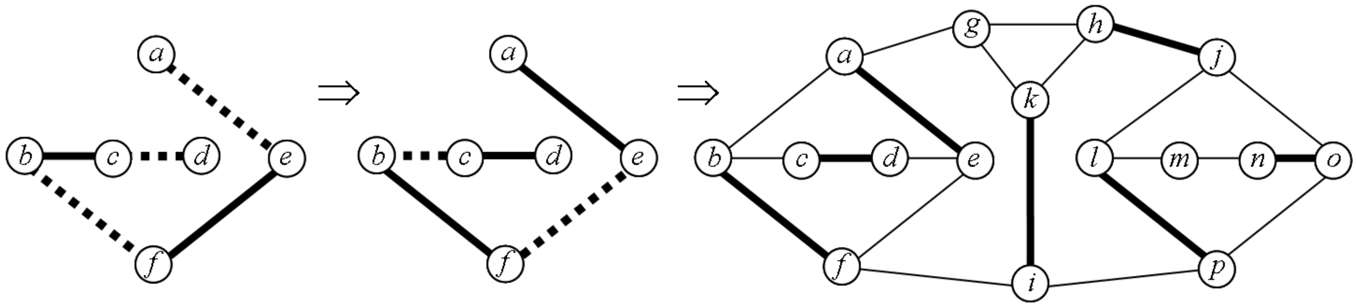
(iii) Neither matching is perfect since G has 16 vertices, so a perfect matching will have $16/2 = 8$ edges, which is larger than M and M'.

(b) (i) $M' \Delta M$ contains the edges that are in M or M' but not both. This is shown below (combined with (ii)).

(ii) Diagrammatically:



(iii) (combined with (iv)



(c) An augmenting path for the increased matching is: gh, hj, jo, no, mn. Exchanging matching and non-matching edges on this path gives the following perfect matching:

$$\{ae,\ bf,\ cd,\ gh,\ ik,\ jo,\ lp,\ mn\}.$$

A minimum cover for G will have cardinality 9. This is not easy to verify (I did a computer check) since we have no efficient algorithm for finding minimum covers in non-bipartite graphs. We could develop some ad-hoc argument based on the fact that triangles need 2 vertices to cover their edges while 5-cycles need 3 vertices; there is one triangle and four 5-cycles, so we start with a requirement of a total of $2+4\times 3 = 14$ vertices but observe that some vertices are shared between cycles. Continuing along this line we can presumably eliminate any possible cover of size 8.

There are just four possible covers of size 9:

$\{b,c,e,g,h,i,l,m,o\}$ $\{b,c,e,g,h,i,l,n,o\}$ $\{b,d,e,g,h,i,l,m,o\}$ $\{b,d,e,g,h,i,l,n,o\}$

We observe that min cover size is strictly greater than max matching size.