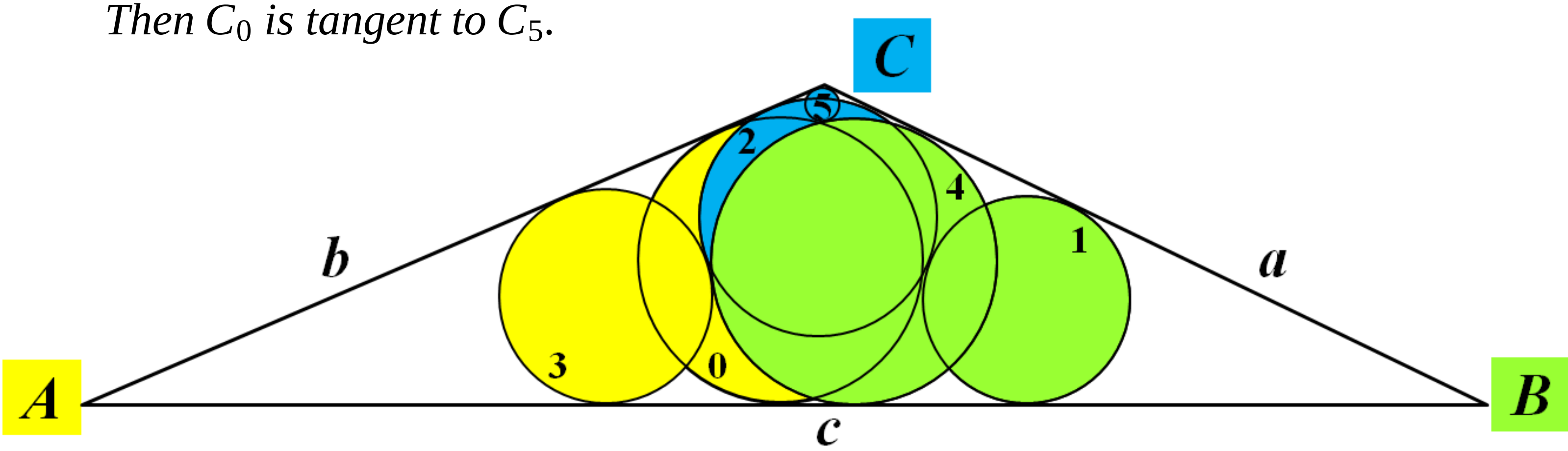
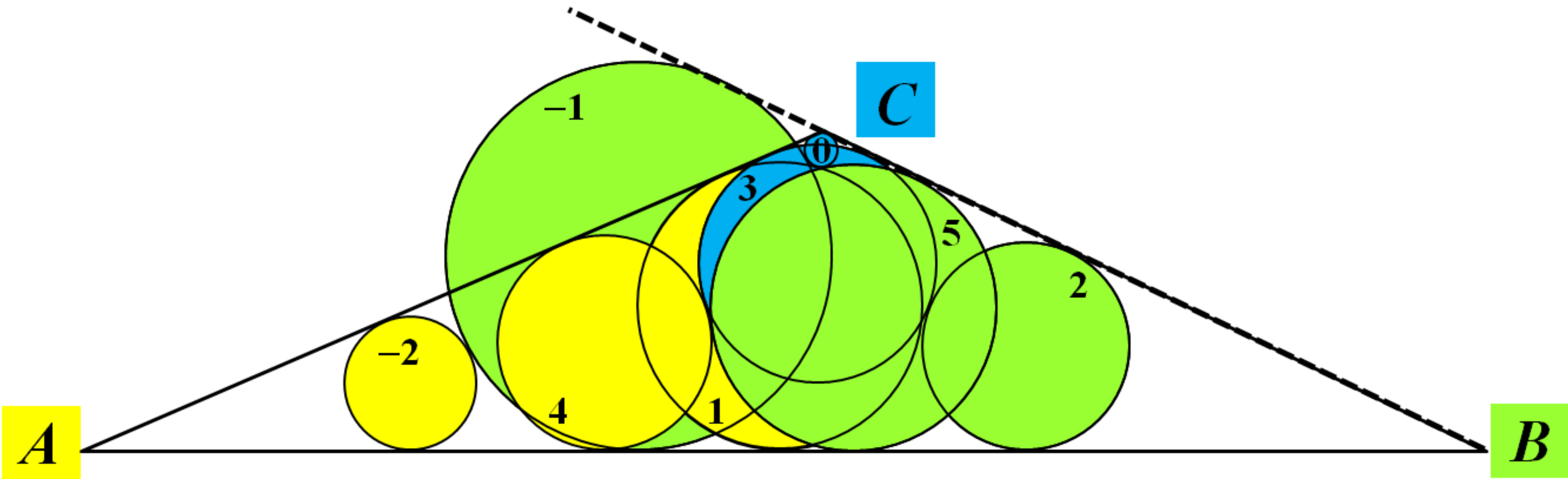
THEOREM OF THE DAY

The Six-Circles Theorem *Consider a triangle on sides a, b and c. Construct six circles,* $C_0, \ldots, C_5$*, in the triangle, with the tangencies shown in the following table, chosen so that the centre of each circle* C_i*,* $i \geq 1$*, is closest to the vertex incident with its tangent sides:*

C_0	C_1	C_2	C_3	C_4	C_5
b, c	a, c, C_0	a, b, C_1	b, c, C_2	a, c, C_3	a, b, C_4

Then C_0 *is tangent to* C_5*.*





The condition on the centres of the circles resolves an ambiguity: there are two circles tangent to another and to two sides. On the left, for example, the circles labelled '0' and '1' are tangent. But a larger circle tangent to edges *a* and *c* can be chosen, tangent to circle '0' on the opposite side of its centre. This choice is excluded.

The theorem asserts that circles constructed as specified form a 'period' of length 6, after which the construction must repeat. This is illustrated on the left, with the whole construction being predetermined once circle '0' is chosen.

Various adaptations and generalisations are possible. In the construction shown bottom left, a circle tangent to circle '−2' and sides *a* and *c* is possible only if triangle sides may be extended, as indicated by the dashed line. In this case D. Ivanov and S. Tabachnikov have shown that the construction is again periodic with period 6, but with a lead-in or 'pre-period' whose length may be arbitrarily long. Here, circles '−2' and '−1' precede the length 6 period on circles '0' to '5'.

This gem was discovered by J.A. Tyrrell collaborating with amateurs C.J.A. Evelyn and G.B. Money-Coutts. Tyrrell and his student M.T. Powell published a proof in 1971, of a stronger 9-circle version replacing triangle sides a, b, c with arcs of three circles in general position.

Web link: arxiv.org/abs/1312.5260; a nice app is given here: blog.janmr.com/2014/04/six-circles-theorem-illustrated.html.

Further reading: *Mathematical Omnibus: 30 Lectures on Classic Mathematics*, by D. Fuchs and S. Tabachnikov, AMS, 2007.