Theorem of the Day

The Analyst’s Travelling Salesman Theorem Define a dyadic cube, \( Q \subset \mathbb{R}^n \), to be a Cartesian product \( Q = \prod_{i=1}^{n} [m_i 2^{-k}, (m_i + 1)2^{-k}] \), of closed intervals of length \( l(Q) = 2^{-k} \), \( m_i, k \in \mathbb{Z}, k \geq 0 \). Let \( 3Q \) denote the cube having the same centroid as \( Q \) and having side length \( 3l(Q) \), and for \( K \) a bounded subset of \( \mathbb{R}^n \), let \( r_K(Q) \) denote the minimum radius of any cylinder enclosing \( K \cap 3Q \). Now set \( \beta_K(Q) = r_K(Q)/l(Q) \). Then \( K \) is contained in some rectifiable curve if and only if

\[ \sum_Q \beta_K^2(Q)l(Q) < \infty, \]

where the sum is over all dyadic cubes.

The operation of the theorem is shown schematically on the right. Every point in \( n \)-dimensional space lies in an infinite sequence of dyadic cubes, each cube \( Q \) having side-length, \( l(Q) \), half that of the preceding cube in the sequence. Now expand each cube to \( 3Q \) and consider its intersection with some bounded set \( K \). For each \( Q \), the value \( \beta_K(Q) = r_K(Q)/l(Q) \) is a measure of how far \( K \) deviates from a straight line, in the vicinity of \( Q \) and scaled by \( l(Q) \). Choose, by angling as convenient, the thinnest cylinder which will contain this deviation. The principle of the theorem is that, for a ‘well-behaved’ set of points, \( K \), decreasing dyadic cubes will quickly enclose \( K \) within an infinitesimally thin cylinder, allowing the sum in the theorem to converge.

The set \( K \) shown on the right is the famous ‘Koch snowflake’ which is bounded and continuous but not rectifiable (because it has infinite length). Even within an infinitesimally small \( 3Q \), the snowflake exhibits the same complex structure as seen here — the dyadic cubes cannot tame it and the summation over all of them will be infinite.

The traditional ‘travelling salesman’ must tour as economically as possible a finite set of cities. The publication of this continuous (‘analyst’s’) version in \( \mathbb{R}^2 \) by Peter Jones of Yale University in 1990 was followed in 1992 by a proof for arbitrary \( n \) by Kate Okikiolu, then at Princeton.

Web link: arxiv.org/abs/cs/0512042 proves a ‘computable’ version of the theorem and provides an excellent account of the mathematical background.