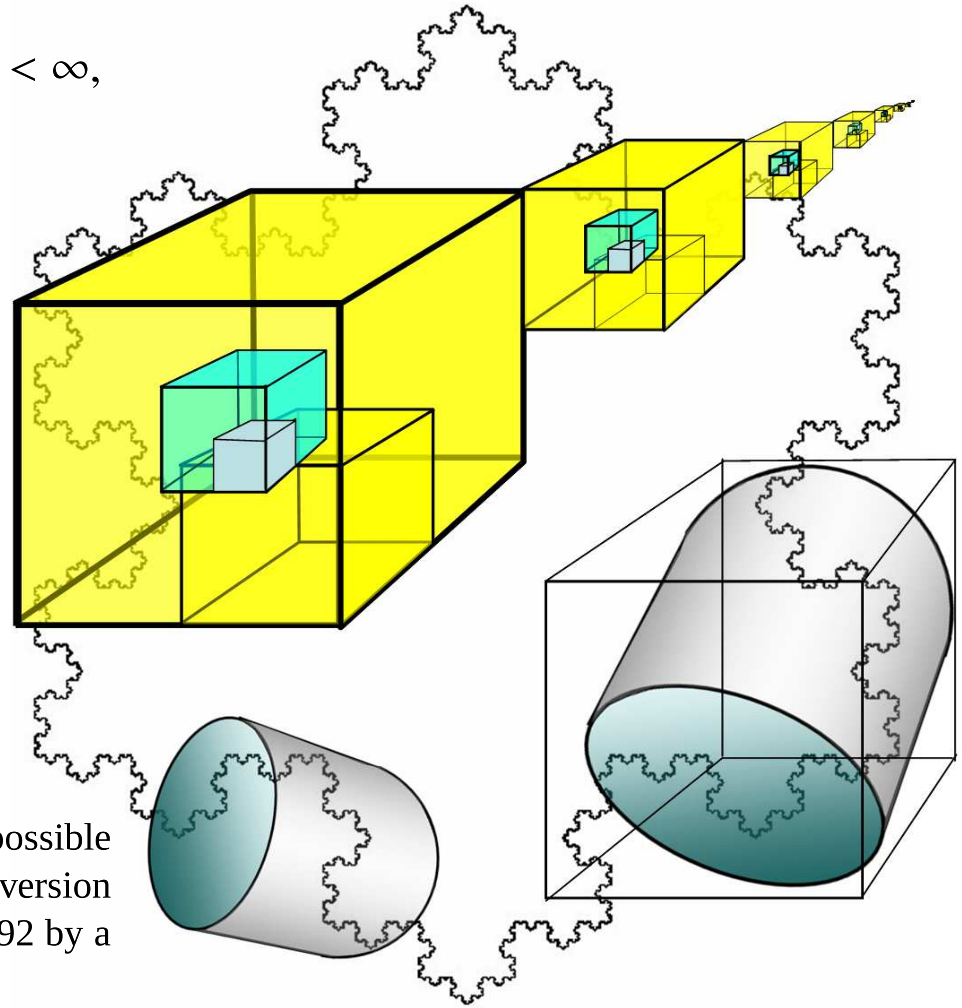
THEOREM OF THE DAY

The Analyst's Travelling Salesman Theorem *Define a* dyadic cube, $Q \subset \mathbb{R}^n$, *to be a Cartesian product* $Q = \prod_{i=1}^{n}[m_i 2^{-k}, (m_i+1)2^{-k}]$, *of closed intervals of length* $l(Q) = 2^{-k}$, $m_i, k \in \mathbb{Z}, k \geq 0$. *Let* $3Q$ *denote the cube having the same centroid as* Q *and having side length* $3l(Q)$, *and for* K *a bounded subset of* $\mathbb{R}^n$, *let* $r_K(Q)$ *denote the minimum radius of any cylinder enclosing* $K \cap 3Q$. *Now set* $\beta_K(Q) = r_K(Q)/l(Q)$. *Then* K *is contained in some rectifiable curve if and only if*

$$\sum_Q \beta_K^2(Q)l(Q) < \infty,$$

where the sum is over all dyadic cubes.



The operation of the theorem is shown schematically on the right. Every point in n-dimensional space lies in an infinite sequence of dyadic cubes, each cube Q having side-length, $l(Q)$, half that of the preceding cube in the sequence. Now expand each cube to $3Q$ and consider its intersection with some bounded set K. For each Q, the value $\beta_K(Q) = r_K(Q)/l(Q)$ is a measure of how far K deviates from a straight line, in the vicinity of Q and scaled by $l(Q)$. Choose, by angling as convenient, the thinnest cylinder which will contain this deviation. The principle of the theorem is that, for a 'well-behaved' set of points, K, decreasing dyadic cubes will quickly enclose K within an infinitesimally thin cylinder, allowing the sum in the theorem to converge.

The set K shown on the right is the famous 'Koch snowflake' which is bounded and continuous but not rectifiable (because it has infinite length). Even within an infinitesimally small $3Q$, the snowflake exhibits the same complex structure as seen here — the dyadic cubes cannot tame it and the summation over all of them will be infinite.

The traditional 'travelling salesman' must tour as economically as possible a finite set of cities. The publication of this continuous ('analyst's') version in $\mathbb{R}^2$ by Peter Jones of Yale University in 1990 was followed in 1992 by a proof for arbitrary n by Kate Okikiolu, then at Princeton.

Web link: arxiv.org/abs/cs/0512042 proves a 'computable' version of the theorem and provides an excellent account of the mathematical background.

Further reading: *Geometric Measure Theory: a Beginner's Guide* by Frank Morgan, Academic Press, 2000.