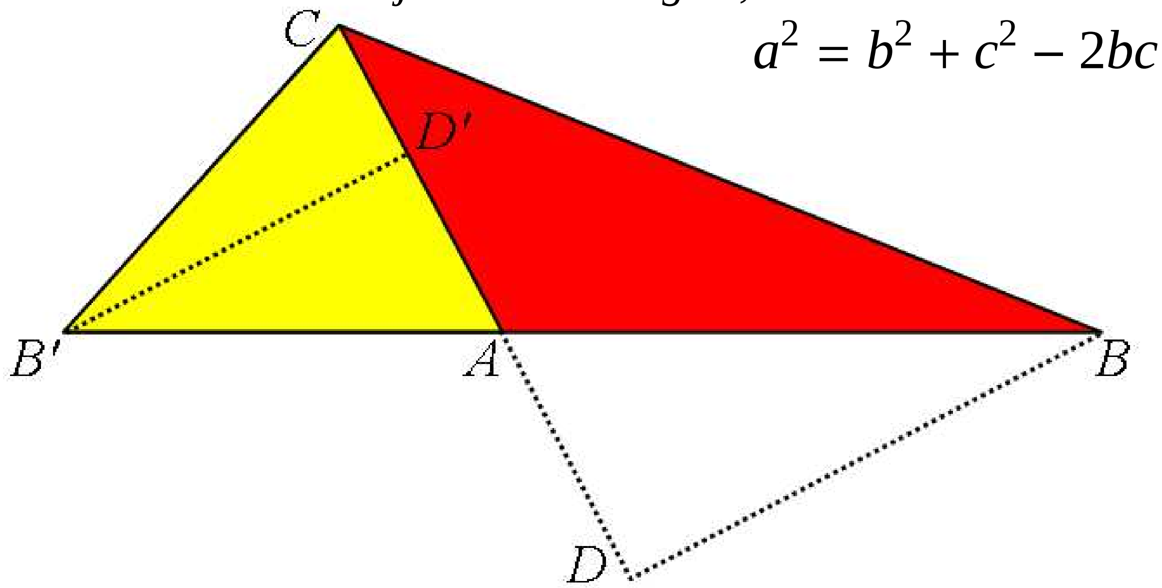
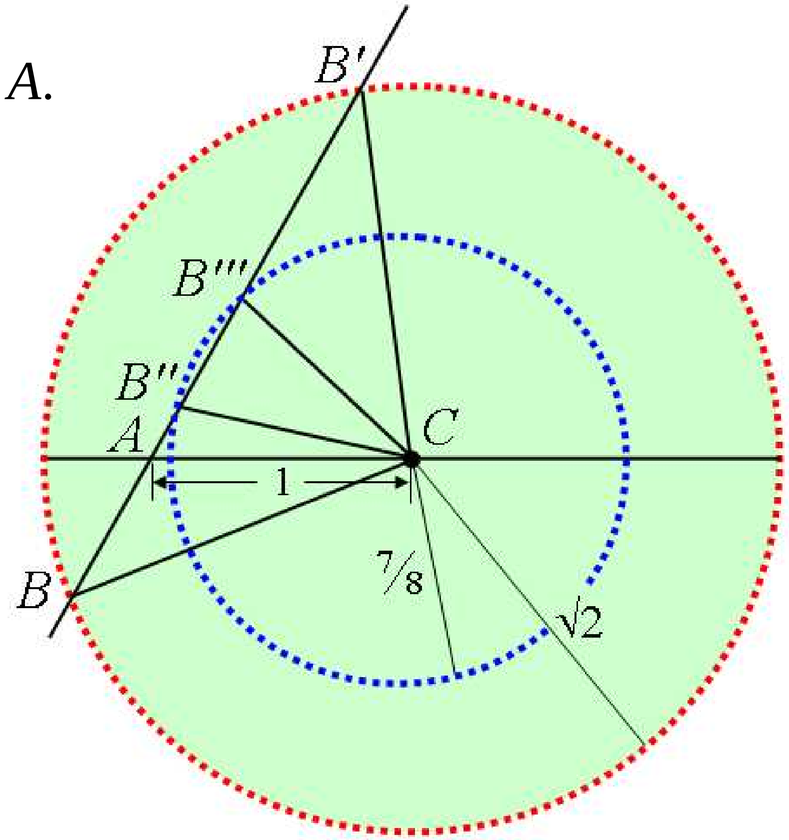
THEOREM OF THE DAY

al-Kāshi's Law of Cosines *If A is the angle at one vertex of a triangle, a is the opposite side length, and b and c are the adjacent side lengths, then*

$$a^2 = b^2 + c^2 - 2bc\cos A.$$





= Euclid Book 2, Propositions 12 and 13 + definition of cosine *If a triangle has vertices A, B and C and side lengths AB, AC and BC, and if the perpendicular through B to the line through A and C meets this line at D, and if the angle at A is obtuse then*

$$BC^2 = AB^2 + AC^2 + 2AC.AD,$$

while if the angle at A is acute then the last term on the right-hand-side is subtracted.

Euclid's two propositions supply the Law of Cosines by observing that $AD = AB\cos(\angle DAB) = AB\cos(180^\circ - \angle CAB) = -AB\cos(\angle CAB)$; while in the acute angle case (shown above left as the triangle on vertices A, B', C), the subtracted length AD' is directly obtained as $AB'\cos(\angle D'AB') = AB'\cos(\angle CAB')$.

The Law of Cosines leads naturally to a quadratic equation, as illustrated above right. Angle $\angle BAC$ is given as 120°; the triangle ABC has base $b = 1$ and opposite side length $a = \sqrt{2}$. What is the side length $c = AB$? We calculate $(\sqrt{2})^2 = 1^2 + c^2 - 2\times 1\times c\cos 120^\circ$, which gives $c^2 + c - 1 = 0$, with solutions $-\varphi$ and $1/\varphi$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The positive solution is the length of side AB; the negative solution corresponds to the 2nd point where a circle of radius $\sqrt{2}$ meets the line through A and B. This is the point B', with the triangle $AB'C$ being the acute angle version of Euclid. If the circle radius is reduced to 7/8 then the quadratic becomes $(8c-3)(8c-5) = 0$ with two positive solutions, corresponding to the points B'' and B''', both triangles belonging to the acute angle case.

The modern trigonometric functions originate nearly 800 years after Euclid in Hindu mathematics of the 5th century. Nearly 1000 years on, Jamshid al-Kāshi (c. 1380–1429) gave the first modern version of the Law of Cosines.

Web link: www.clarku.edu/~djoyce/trig/laws.html

Further reading: *Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook* by Victor Katz (ed.), Princeton University Press, 2007.