THEOREM OF THE DAY

A Theorem on Apollonian Circle Packings For every integral Apollonian circle packing there is a unique ‘minimal’ quadruple of integer curvatures, \((a, b, c, d)\), satisfying \(a \leq 0 \leq b \leq c \leq d\), \(a+b+c+d > 0\) and \(a + b + c \geq d\). This so-called root quadruple completely specifies the packing.

A Descartes configuration consists of four mutually tangent circles. Above right, for example, is a circle of radius 1/7 containing circles of radius 1/12, 1/17 and 1/20, each of which has a point of contact with the other three. The integers labelling the circles are the curvatures (the reciprocals of the radii) and in the root quadruple of curvatures, \((-7, 12, 17, 20)\), the enclosing circle of radius 1/7 is determined to have negative curvature so that all four circles have disjoint interiors. Any such configuration specifies four more tangent circles — above right, these have curvatures 24, 33, 48, and 105, producing four new configurations \((-7, 12, 17, 24)\), \((-7, 12, 20, 33)\), \((-7, 17, 20, 48)\) and \((12, 17, 20, 105)\). Repeating this process produces a system of infinitely packed circles: an Apollonian circle packing. If our initial configuration is integral, as in each of the above examples (which are drawn to different scales), then we will get an integral packing with every curvature an integer.

This theorem comes from a series of four pivotal papers by the AT&T team of Ronald Graham, Jeffrey Lagarias, Colin Mallows and Allan Wilks, together with Catherine Yan of Texas A&M University. They further show that all integral Apollonian circle packings may be derived from root quadruples which, like those depicted above, have entries whose gcd is 1.


A good guide to creating your own circle packings is Cliff Reiter’s article at archive.vector.org.uk/22/4.