THEOREM OF THE DAY

A Theorem on Apollonian Circle Packings For every integral Apollonian circle packing there is a unique ‘minimal’ quadruple of integer curvatures, \((a, b, c, d)\), satisfying \(a \leq 0 \leq b \leq c \leq d\), \(a+b+c+d > 0\) and \(a + b + c \geq d\). This so-called root quadruple completely specifies the packing.

A Descartes configuration consists of four mutually tangent circles. Above, for example, is a circle of radius \(1/7\) containing circles of radius \(1/12, 1/17\) and \(1/20\), each of which has a point of contact with the other three. The integers labelling the circles are the curvatures (the reciprocals of the radii) and in the root quadruple of curvatures, \((-7, 12, 17, 20)\), the enclosing circle of radius \(1/7\) is determined to have negative curvature so that all four circles have disjoint interiors. Any such configuration specifies four more tangent circles — above, these have curvatures \(24, 33, 48,\) and \(105\), producing four new configurations \((-7, 12, 17, 20), (-7, 12, 20, 33), (-7, 17, 20, 48)\) and \((12, 17, 20, 105)\). Repeating this process produces a system of infinitely packed circles: an Apollonian circle packing. If our initial configuration is integral, as in the above example, then we will get an integral packing with every curvature an integer.

This theorem comes from a series of four pivotal papers by the AT&T team of Ronald Graham, Jeffrey Lagarias, Colin Mallows and Allan Wilks, together with Catherine Yan of Texas A&M University. They further show that all integral Apollonian circle packings may be derived from root quadruples which, like that depicted above, have entries whose gcd is 1.

Web link: [www.vector.org.uk/archive/v224/cliff224.htm](http://www.vector.org.uk/archive/v224/cliff224.htm) shows how to create Apollonian packings and links to the papers of Graham et al.


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