



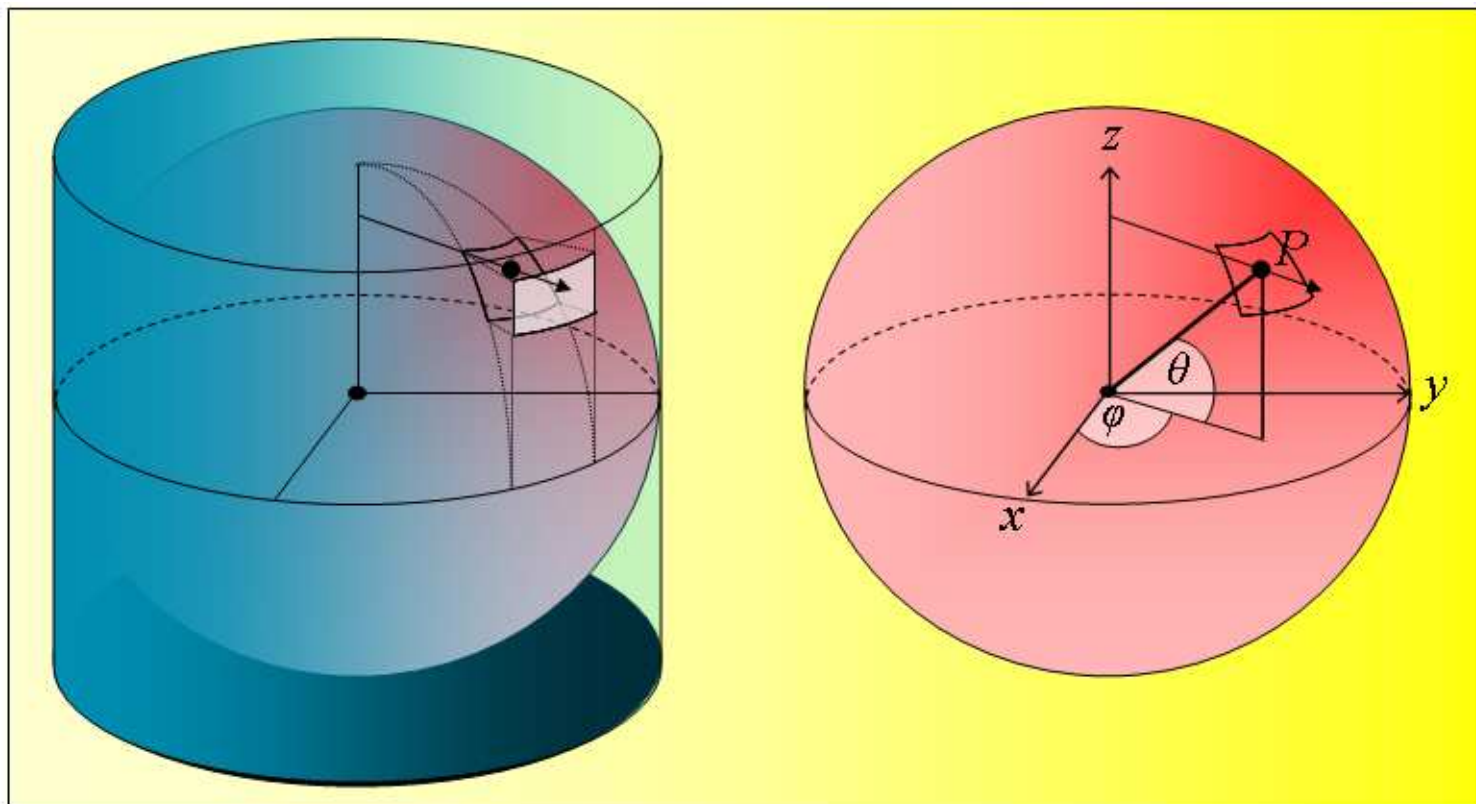
# THEOREM OF THE DAY

**Archimedes' Equiareal Map Theorem** Let  $S$  be the unit sphere defined relative to the  $x$ ,  $y$  and  $z$  axes by  $x^2 + y^2 + z^2 = 1$  and let  $S'$  be the bounding cylinder defined by  $x^2 + y^2 = 1$ . Let  $f$  be the map

$$f : (x, y, z) \mapsto \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z \right),$$

from  $S \setminus \{(0, 0, 1), (0, 0, -1)\}$  to  $S'$  which takes a point  $P$  on the sphere having height  $z_P$ ,  $|z_P| < 1$ , to the nearest point on the cylinder which has height  $z_P$  and which is collinear with  $P$  and  $(0, 0, z_P)$ . Then  $f$  is an equiareal diffeomorphism.

Roughly speaking, Archimedes' map  $f$  is *equiareal* because it maps regions in  $S$  to regions in  $S'$  of the same area. In particular, it maps the whole surface of the sphere to the cylindrical surface whose area is equal to height  $\times$  cylinder circumference  $= 2\pi$  ( $\tau = 2\pi$ ). The map is a *diffeomorphism* because it is differentiable and has a differentiable inverse; this is a technical condition which allows the machinery of differential geometry to be brought to bear on the problem. And nowadays, it is to differential geometry that Archimedes' theorem naturally belongs. Our sphere  $S$  is a 2-dimensional surface by virtue of a 2-variable parameterisation, as shown, far right:  $\theta$  measures latitude;  $\varphi$  measures longitude. The coordinates of point  $P$  are calculated as  $\sigma(\theta, \varphi) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta)$ . We compute lengths, angles and areas on  $S$  using the *first fundamental form*, defined in terms of three scalar products of partial derivatives:



$$E = \sigma_\theta \cdot \sigma_\theta, \quad F = \sigma_\theta \cdot \sigma_\varphi, \quad G = \sigma_\varphi \cdot \sigma_\varphi.$$

For the sphere  $S$  we have  $E = 1$ ,  $F = 0$ ,  $G = \cos^2 \theta$ ; for the cylinder  $S'$ , parameterised, under  $f$ , by  $\sigma'(\theta, \varphi) = (\cos \varphi, \sin \varphi, \sin \theta)$ , we get  $E' = \cos^2 \theta$ ,  $F' = 0$ ,  $G' = 1$ . Now there is a theorem of differential geometry which says that  $f$  will be equiareal if and only if  $EG - F^2 = E'G' - F'^2$ . Since this is true, so too is the theorem of Archimedes.

This theorem links the finest achievements of (maybe) the three finest mathematicians of all time: from Archimedes' discoveries on spherical measurement, c. 225BC, via the calculus of Newton, c. 1665, to the invention of differential geometry by Gauss, c. 1825.

**Web link:** [people.maths.ox.ac.uk/hitchin/hitchinnotes/hitchinnotes.html](http://people.maths.ox.ac.uk/hitchin/hitchinnotes/hitchinnotes.html): Chapter 3 of Geometry of Surfaces

**Further reading:** *Elementary Differential Geometry* by Andrew N. Pressley, Springer, London, 2nd edition, 2010, chapter 6.

