THEOREM OF THE DAY

The Art-Gallery Theorem Let $P$ be the subset of the Euclidean plane consisting of an $n$-vertex simple polygon and its interior. Then $P$ contains a finite subset $S$, of cardinality at most $\left\lfloor \frac{n}{3} \right\rfloor$, such that every point of $P$ is joined to some point of $S$ by a straight line contained in $P$.

The examples on the left are based on Steve Fisk’s beautiful 1978 proof of this theorem. If a triangulation of the polygon is properly 3-coloured (vertices coloured so that no edge joins the same colours) then any smallest colour class of vertices may form our set $S$. Note that ‘triangulation’ means adding edges between vertices so that every point of $P$ belongs to a triangle having precisely three polygon vertices. Thus, at ❶, although $P$ is divided up into triangles, the triangulation is incomplete; and although the 3-colouring is valid it does not guarantee a valid set $S$: no vertex from the smallest colour class, the red vertices, can ‘see’ the point in $P$ marked ‘X’. At ❷, a complete triangulation is given. The 3-colouring is produced systematically by joining the triangles into a tree, as shown, and then traversing the tree ‘depth first’, 3-colouring triangle by triangle. And indeed each colour class is a valid candidate for set $S$ and has cardinality $\left\lfloor \frac{12}{3} \right\rfloor = 4$. However, not all triangulations are equal! The one at ❸ produces a set $S$ (the blue vertices) which is optimal, having cardinality 2. So we can sometimes do better than $\left\lfloor \frac{n}{3} \right\rfloor$; but not always—in the example at ❹ each triangle necessarily adds an extra point to set $S$.

This theorem was published by Václav Chvátal in 1975 in response to a question by Victor Klee. The lower bound can, in the words of Chvátal’s original paper “be interpreted as the minimum number of guards required to supervise any art gallery with $n$ walls.”


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