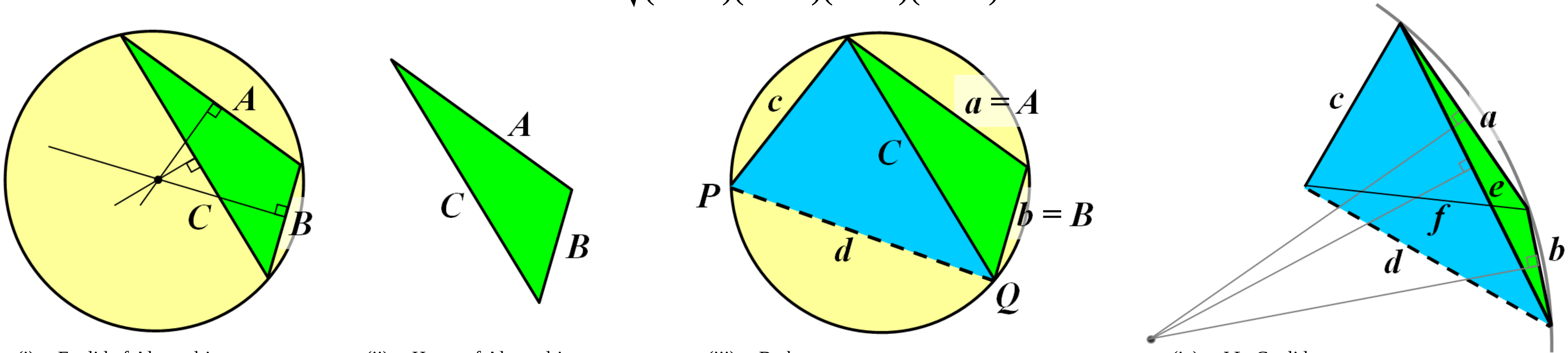
THEOREM OF THE DAY

Brahmagupta's Formula *The area K of a cyclic quadrilateral with side lengths a, b, c, d and semiperimeter $s = (a+b+c+d)/2$ is given by*

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$



(i) Euclid of Alexandria
Elements, Book IV, (*c.* 300 BC)
Circumcircling the triangle

(ii) Heron of Alexandria
Metrica, (1st century AD)
Sides → area (triangle)

(iii) Brahmagupta
The Brāhmasphutasiddhānta, (628 AD)
Sides → area (cyclic quadrilateral)

(iv) J.L. Coolidge
Amer. Math. Monthly, (1939)
Sides → area (any quadrilateral)

(i) The familiar formula for triangular area ($1/2 \times$ base $\times$ height) was known to Greek mathematicians for at least three hundred years before Euclid catalogued all of known geometry in his *Elements*, including (Book IV, Proposition 5) the construction of the unique *circumcircle* which passes through the vertices of a triangle, by intersecting the perpedicular bisectors of its sides.

(ii) Three hundred years after *that* came the famous **Heron's formula**: *the area K of a triangle with sides A, B, C and semiperimeter $s = (A+B+C)/2$ is given by $K = \sqrt{(s-A)(s-B)(s-C)s}$.*

(iii) Although Greek mathematics was apparently unknown to medieval Indian (as opposed to Islamic) scholars, Brahmagupta effectively put Heron's triangle back into the circle: take two non-overlapping circumscribed triangles sharing a common edge (C in the picture); the result is a *cyclic quadrilateral*, one whose vertices all lie on a circle. And now the area of the quadrilateral replaces the final s in Heron's formula by $s-d$. If point P is allowed to approach point Q then d becomes zero and c becomes C, recovering Heron.

(iv) Another thirteen hundred years pass and the circumscribing circle is removed once more in American mathematician Julian Lowell Coolidge's **quadrilateral area formula:** *the area of an arbitrary convex quadrilateral with sides a, b, c, d, a opposite to d, with diagonals e, f, and with semiperimeter s, is given by $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{4}(ad+bc+ef)(ad+bc-ef)}$.* This generalises Brahmagupta by virtue of another classic of antiquity, **Ptolemy's Theorem:** *quadrilateral a, b, c, d, a opposite to d, with diagonals e, f, is cyclic if and only if $ad + bc = ef$.*

Brahmagupta's formula appears in his *Brāhmasphutasiddhānta*, a treatise on astronomy. Brahmagupta's writings contain the first known treatment of zero and negative numbers.

Web link: www.mathpages.com/home/kmath196/kmath196.htm

Further reading: *Journey Through Genius* by William Dunham, John Wiley & Sons, 1990.