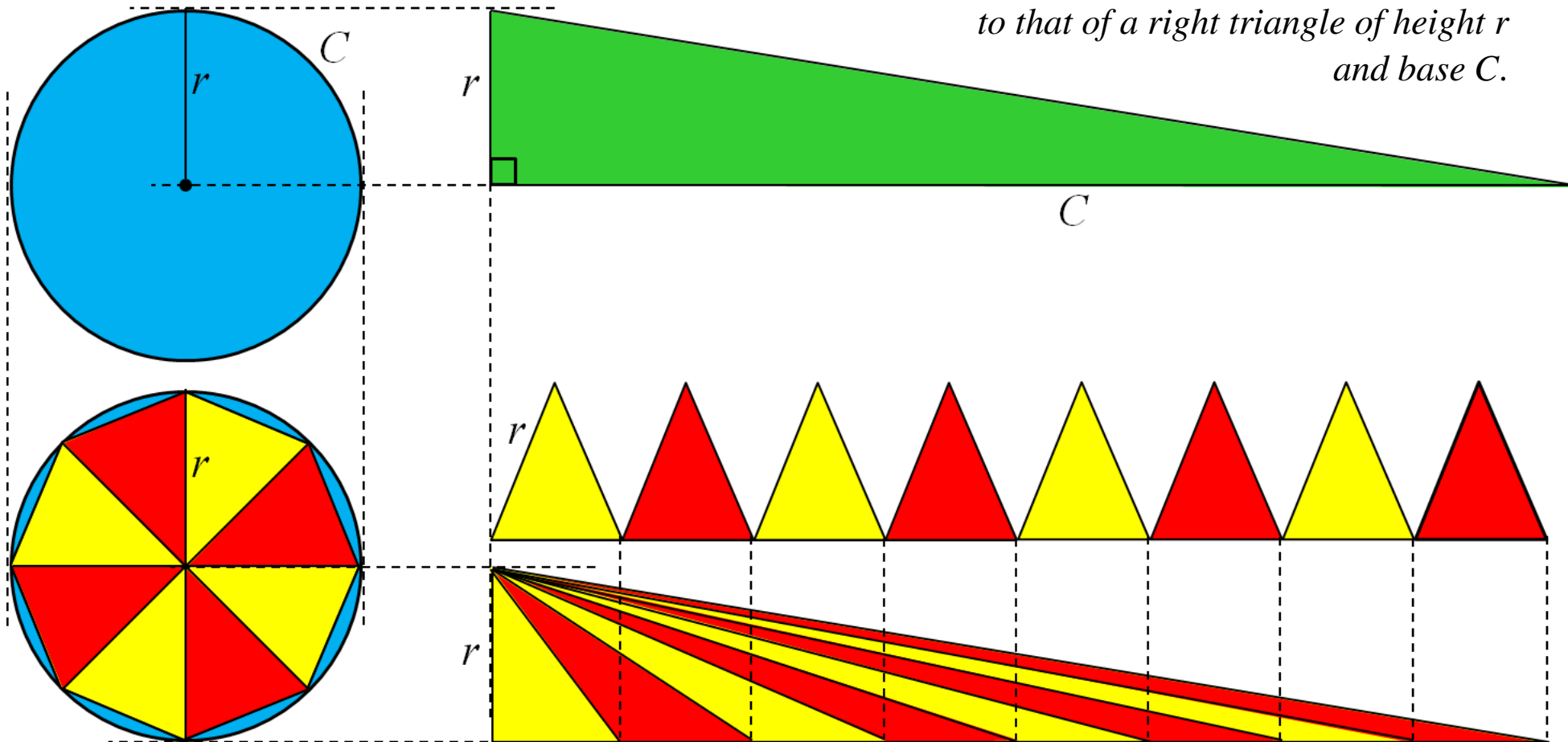




# THEOREM OF THE DAY

**The Circle Area Theorem** *The area of a circle of radius  $r$  and circumference  $C$  is identical to that of a right triangle of height  $r$  and base  $C$ .*



Ancient Greek mathematics had no notion of ‘formula’: facts about geometrical measure were stated as equalities involving different shapes. The Greek’s ‘method of exhaustion’ proved such equalities by showing that some infinite construction forced the equality. The equivalence of circle area  $A_C$  and triangle area  $A_T$  was thus proved by showing that  $A_C \leq A_T$  and then showing that  $A_C < A_T$  was impossible. The infinite construction is the engine of the method. Here a regular polygon is inscribed in our circle of radius  $r$  and is triangulated. The triangles remain ‘equiareal’ provided height and base are preserved and are thus formed into a right triangle whose area is identical to that of the regular polygon. And now, as the number of sides of the polygon increase, ‘exhausting’ the circle area, the base of our right triangle approaches  $C$ . With modern algebra we express the conclusion as  $A_C = A_T = \frac{1}{2}Cr$  and calculate  $\frac{1}{2}Cr = \frac{1}{2}r\tau \times r = \frac{1}{2}\tau r^2$ , recognising the result as the special case  $\theta = \tau$  of the area formula  $\frac{1}{2}r^2\theta$  for a sector subtended by angle  $\theta$ .

Archimedes (3rd century BC) established his circle area equality in an essay ‘Measurement of a circle’ in which he also famously used infinite sequences of inscribed and circumscribed polygons to approximate the ratio of circumference to diameter. Euclid’s proof of his Proposition 2, Book XII (c.300BC) is a similar application of the method of exhaustion but Archimedes attributes the method to Eudoxus a century earlier.

**Web link:** [www.ams.org/samplings/feature-column/fc-2012-02](http://www.ams.org/samplings/feature-column/fc-2012-02)

**Further reading:** *The Works of Archimedes* edited by Thomas Little Heath, CUP reprint, 2009.

