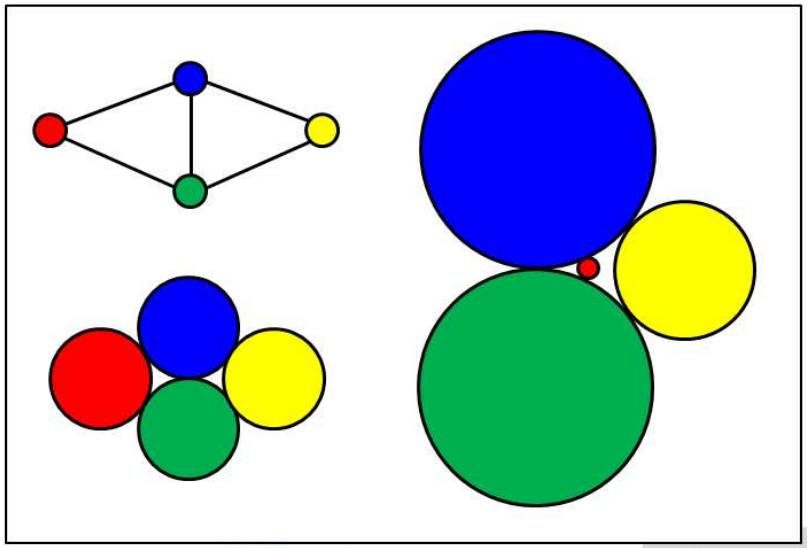
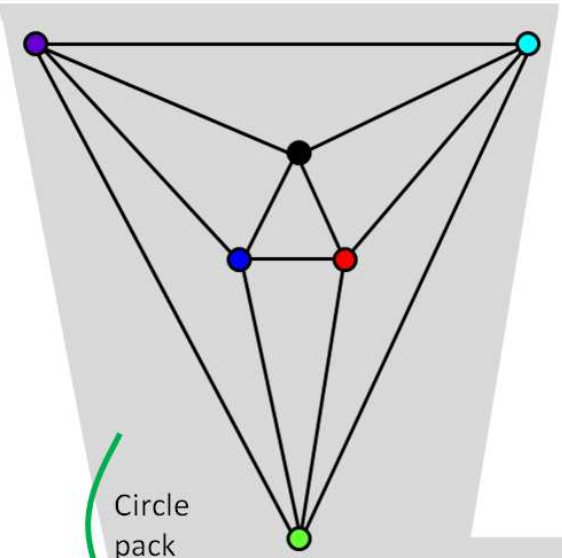
THEOREM OF THE DAY

The Circle Packing Theorem *For any simple planar graph G there exists a circle packing in the complex plane whose tangency graph is isomorphic to G. Moreover, if G is maximally planar then this packing is unique up to Möbius transformations.*

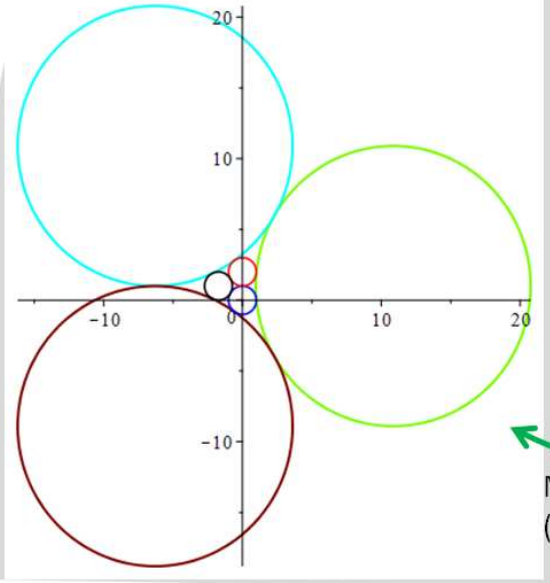




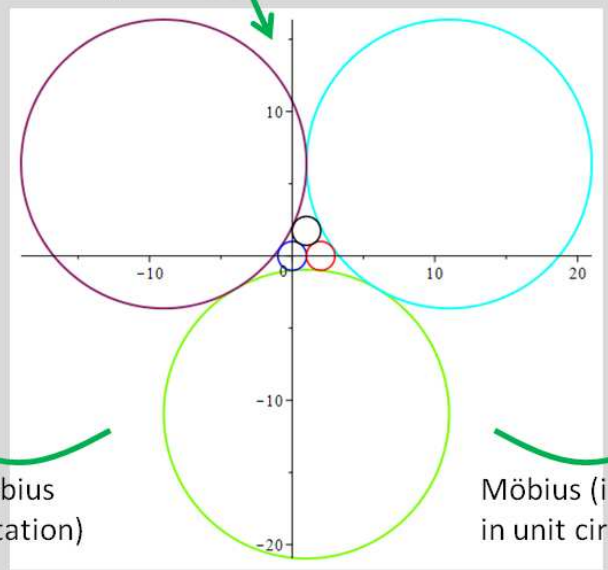
Circle pack

For the 4-vertex graph far-left, the edges correspond to the tangencies among four circles in the packing below it. But not uniquely, since the circle packing to its right represents the same correspondence. The 6-vertex graph near-left, however, is maximally planar: any additional edges, however we redraw the graph, will cause at least two edges to intersect each other. A circle packing for this graph is plotted (in the complex plane) directly below it. This plot is 'essentially unique': it may be translated in the plane or, as shown below left, rotated; or it may be mirrored across the boundary of a circle, as shown below right. These transformations, together with scaling, are precisely those offered by the Möbius function $f(z) = \frac{az+b}{cz+d}$, for complex values a, b, c, d and with $ad - bc \neq 0$ (or else $f(z) = \frac{a}{c}$, a constant).

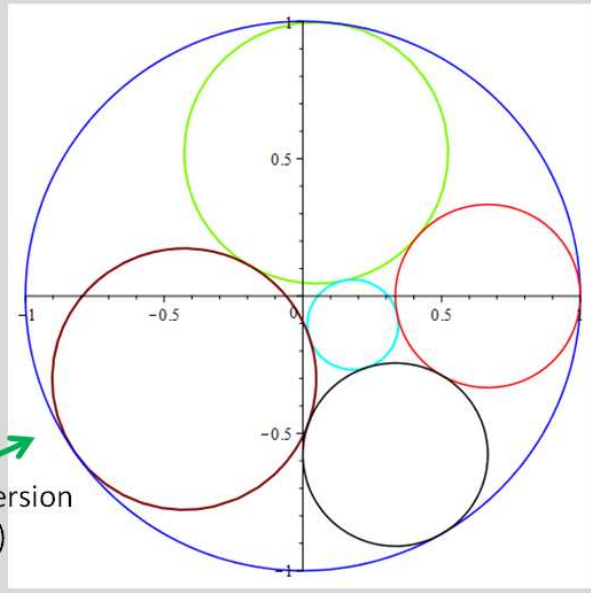
Thus, in our example, the anticlockwise quarter-turn rotation (far left) is specified by setting $a = i, b = c = 0, d = 1$. The reflection (near left) into the unit circle is $a = 0, b = c = 1, d = 0$.



Möbius (rotation)



Möbius (inversion in unit circle)



The existence theorem for circle packings was discovered in 1936 by the German analyst Paul Koebe who used it in his generalisation of the Riemann Mapping Theorem. Existence and uniqueness were derived independently, forty years later, by the American geometer William Thurston, as a corollary to a 1970 theorem of E.M. Andreev, a student of the Russian algebraist Ernest Vinberg.

Web link: arxiv.org/abs/1812.11224

Further reading: *Circles, Spheres and Spherical Geometry* by Hiroshi Maehara and Horst Martini, Birkhäuser, 2024.