THEOREM OF THE DAY

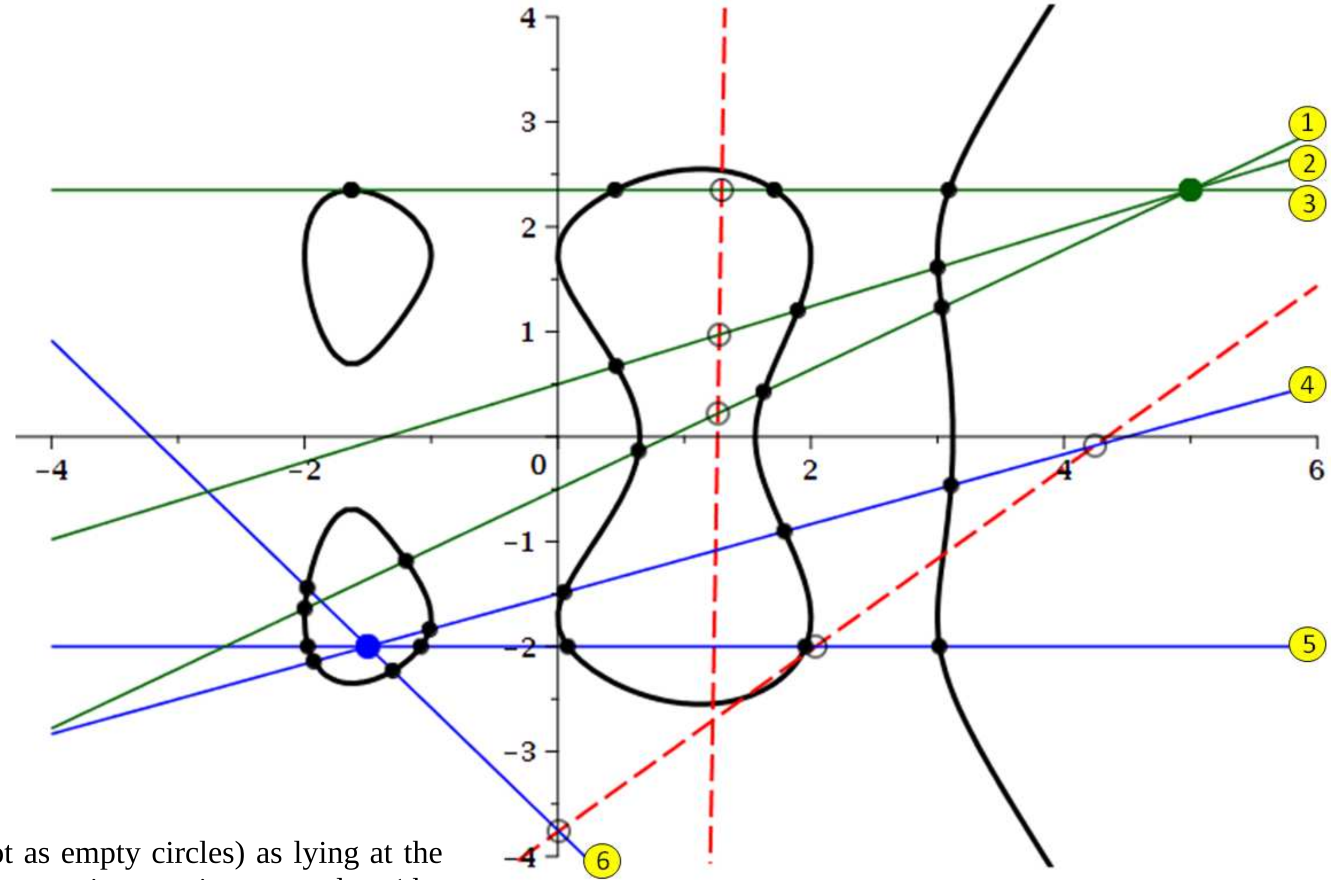
Cotes' Harmonic Means Theorem in Geometry *In the plane, let $f(x,y) = 0$ be an algebraic curve of degree d, denoted by C, and let p be a point not on C. For a straight line L through p, and a second point q on L, denote by pq the distance from p to q, signed + or − consistent with some fixed orientation of L. Let $r_1, \ldots, r_d$ be the d points of intersection of L with C and let p_L be the point on L satisfying*

$$\frac{d}{pp_L} = \frac{1}{pr_1} + \ldots + \frac{1}{pr_d}. \qquad (1)$$

Then the following subset of the plane is a straight line:

$$\cup_L \{p_L \mid p_L \text{ satisfies equation (1)}\},$$

the union being taken over all straight lines L through p.



Our illustration shows a curve C of degree 5 specified as $f(x,y) = 0$, where

$$f(x,y) = \left(y^2 - 3\right)^2 - x(x+1)(x+2)(x-2)(x-3),$$

a variant of the so-called stirrup curve. We have plotted two points with three straight lines through each, numbered 1 to 6, oriented, say, from left to right. Lines 1 and 3 intersect C in five points, line 3 having a double intersection at a point of tangency. Line 2 appears to have only three intersections but the equation that they solve also has two conjugate complex solutions and these work perfectly well in equation (1). This equation calculates the p_L (shown on the plot as empty circles) as lying at the harmonic mean of the distances from p to the r_i, and Cotes' theorem is sometimes stated as 'the harmonic means lie on a straight line'. And indeed, our values of p_L for the two choices of point p lie on two straight lines (shown as dashed lines in the plot). But the fact that lines 4 to 6 exhibit some negative distances to their points of intersection militates against the p_L distances being viewed as some kind of 'average'! (By the way, line 6 does actually meet C in a third point, but at approx. (8.87, −14.09), well outside our plot).

Roger Cotes (1682–1716) is remembered for his significant contributions to the publication of the 2nd edition of Newton's *Principia*. His own mathematical discoveries were published posthumously after his early death.

Web link: www.mathpages.com/home/ihistory.htm: click on *Harmonia Mensurarum*

Further reading: *Metric Algebraic Geometry* by Paul Breiding, Kathlén Kohn and Bernd Sturmfels, Birkhäuser, 2024 (our presentation follows chapter 1).