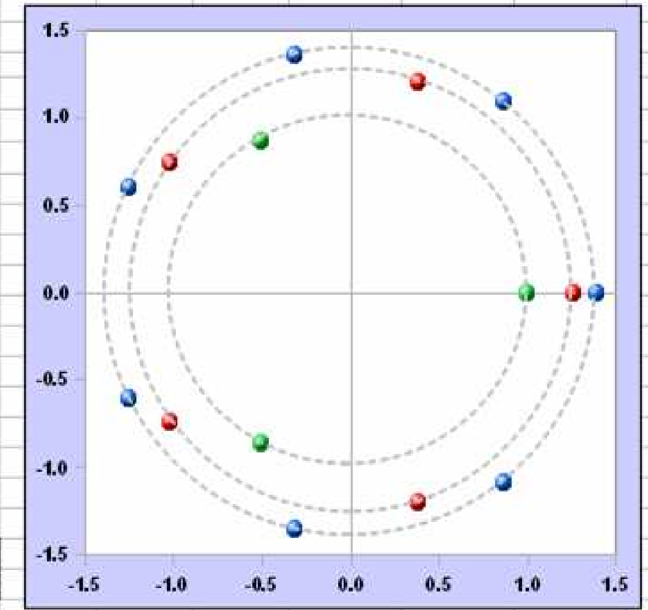
THEOREM OF THE DAY

De Moivre's Theorem *Let θ be an angle and n a positive integer. Then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

F18 {=MMULT(ROOT3;E18:E19)}

	A	B	C	D	E	F	G	H	I	J	K
3											
4											
5		Rotation by Tau/n (anticlockwise)									
6		(Tau = 2*Pi)			ROOT3	(n=3)					
7		=Cos(2Pi/n)	=-Sin(2Pi/n)		-0.5000	-0.866					
8		=Sin(2Pi/n)	=Cos(2Pi/n)		0.8660	-0.500					
9											
10					ROOT5	(n=5)					
11					0.3090	-0.951					
12					0.9511	0.309					
13											
14					ROOT7	(n=7)					
15					0.6235	-0.782					
16					0.7818	0.623					
17		nth roots of									
18		1	real	1.000	-0.500	-0.500	1.000				
19		n=3	complex	0.000	0.866	-0.866	0.000				
20											
21		Pi	real	1.257	0.389	-1.017	-1.017	0.389	1.257		
22		n=5	complex	0.000	1.196	0.739	-0.739	-1.196	0.000		
23											
24		10	real	1.389	0.866	-0.309	-1.252	-1.252	-0.309	0.866	1.389
25		n=7	complex	0.000	1.086	1.355	0.603	-0.603	-1.355	-1.086	0.000



Among other things, De Moivre's theorem is perfect for finding n-th roots of real or complex numbers, as shown in the above screenshot from OpenOffice Calc. In the plot we think of the horizontal axis as recording the real part and the vertical axis the imaginary part of the complex number $z = a + ib$. This number may be written as $z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{a^2+b^2}$ and θ is the angle made with the horizontal axis by the line from $(0,0)$ to (a,b): $\cos\theta = a/r$. On our horizontal axis three real numbers are plotted: $\sqrt[3]{1} = 1$, $\sqrt[5]{\tau/2} \approx 1.257$ ($\tau = 2\pi$) and $\sqrt[7]{10} \approx 1.389$. But there are complex n-th roots too! For instance, the complex number z will be a cube root of 1 if $1 = z^3 = r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta)$. This will happen whenever 3θ is a multiple of τ because $\cos\tau + i\sin\tau = 1 + i\times 0 = 1$. So in our plot we have the point $(1,0)$ and this point rotated by $\tau/3$ and by $2\tau/3$ to give the three cube roots of 1. A nice way to rotate a point (x,y) by an angle τ/n is to multiply the point by the appropriate *rotation matrix* as shown above left. Note that $n = 2$ gives a rotation of the real square root of a number x by $\tau/2$, or 180°, to give $-x$.

Abraham de Moivre discovered a version of his formula in 1707 and proposed the more usual version in 1722. Leonhard Euler's famous 1749 identity $e^{i\theta} = \cos\theta + i\sin\theta$ generalised it to real number exponents.

Web link: www.cimt.org.uk/projects/mepres/alevel/alevel.htm, Ch. 3 of Further Pure Mathematics.

Further reading: *Trigonometric Delights* by Eli Maor, Princeton University Press, 2002