



# THEOREM OF THE DAY

**The Descartes Circle Theorem** *If four circles forming a Descartes configuration have respective curvatures  $b_1, b_2, b_3$  and  $b_4$ , then*

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

For pairs of lips to kiss maybe  
Involves no trigonometry.  
'Tis not so when four circles kiss  
Each one the other three.  
To bring this off the four must be  
As three in one or one in three.  
If one in three, beyond a doubt  
Each gets three kisses from without.  
If three in one, then is that one  
Thrice kissed internally.

Four circles to the kissing come.  
The smaller are the benter.  
The bend is just the inverse of  
The distance from the centre.  
Though their intrigue left Euclid dumb  
There's now no need for rule of thumb.  
Since zero bend's a dead straight line  
And concave bends have minus sign,  
The sum of the squares of all four bends  
Is half the square of their sum.

To spy out spherical affairs  
An oscular surveyor  
Might find the task laborious,  
The sphere is much the gayer,  
And now besides the pair of pairs  
A fifth sphere in the kissing shares.  
Yet, signs and zero as before,  
For each to kiss the other four  
The square of the sum of all five bends  
Is thrice the sum of their squares.

**"The Kiss Precise"**  
Frederick Soddy  
*Nature*, 137, 1936.

A Descartes configuration of circles consists of four mutually tangent circles in the plane. Descartes was concerned with configurations such as that at A in the above display, with bend, or curvature, defined as the reciprocal of radius.

For example, the circles at A might have radii 1/4, 1/12, 1/13 and 1/61.

Then  $(b_1 + b_2 + b_3 + b_4)^2/2 = (4 + 12 + 13 + 61)^2/2 = 8100/2 = 4050$  and  $16 + 144 + 169 + 3721 = 4050$  also, conforming to Descartes' theorem. If we adopt the convention that *negative* curvature corresponds to the interior of a circle being *outside* it, then configuration B is equally valid. For example the circles at B might have radii  $-1/7$  (the outside circle) and 1/12, 1/17 and 1/24 (the inside circles). Then  $(-7 + 12 + 17 + 24)^2/2 = 1058 = 49 + 144 + 289 + 576$  as required. If horizontal lines are taken as circles of infinite radius—"zero bend's a dead straight line"—then the configurations at C and D are again valid. But configuration E is not valid.

As the third stanza of Soddy's poem celebrates, configurations of five spheres in  $\mathbb{R}^3$  can also be made to work, the fraction in the theorem becoming 1/3; with the fraction 1/n, we can go to  $n$  dimensions; and adding curvature of space into the equation, even non-Euclidean Descartes Circle Theorems obtain!

Although he did not prove it completely, this result appears originally in a letter sent by René Descartes to Princess Elisabeth of Bohemia in 1643. There have been many rediscoveries with complete proofs, Jakob Steiner's, in 1826, being perhaps the first. The  $\mathbb{R}^n$  version is due to Thorold Gosset in 1937, with  $n = 3$  being given by Robert Lachlan in 1886.

**Web link:** [arxiv.org/abs/math/0101066](https://arxiv.org/abs/math/0101066). Additions by Gosset and Fred Lunnon to Soddy's poem, which is reproduced courtesy of Nature Publishing Group, can be found here: [www.pballer.net/soddy.html](http://www.pballer.net/soddy.html).

**Further reading:** *Introduction to Circle Packing: The Theory of Discrete Analytic Functions*, by Kenneth Stephenson, CUP, 2005.

