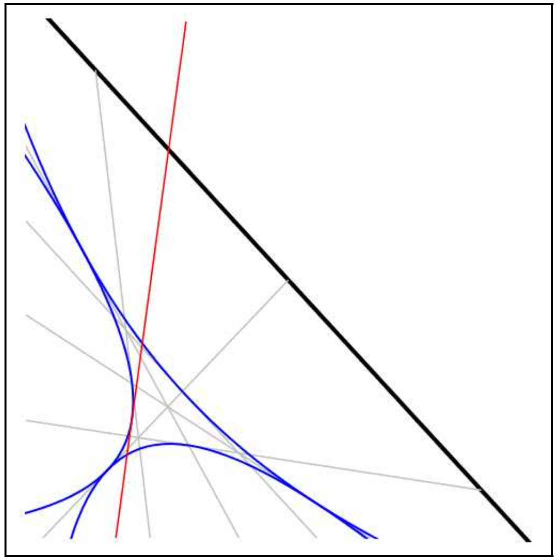
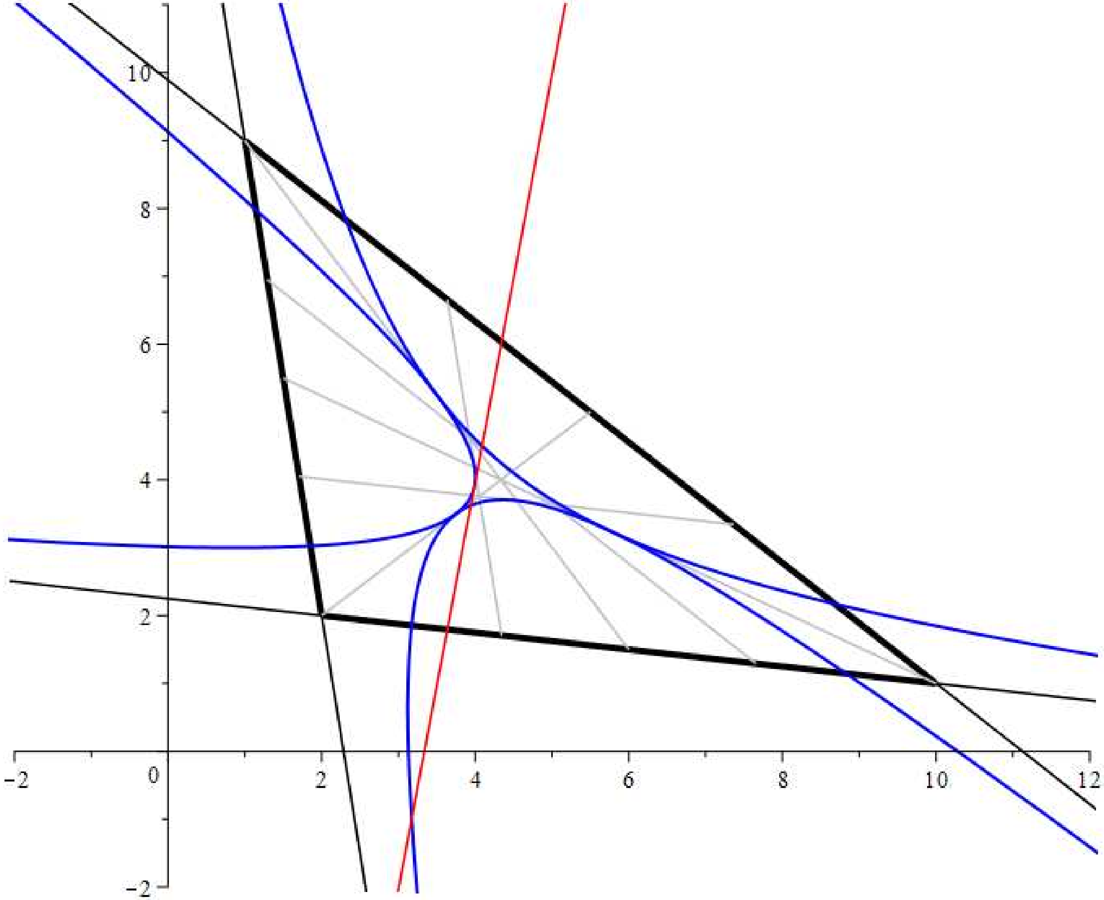
THEOREM OF THE DAY

Dunn and Pretty's Triangle-Halving Deltoid *Let T be a triangle in the Euclidean plane. Then there is a deltoid curve, whose vertices are the intersections of hyperbolae asymptotic to the three pairs of edges of T, such that a given straight line bisects the area of T if and only if it is tangent to the deltoid.*

A **deltoid curve** is formed when three curves meet pairwise in three cusp vertices, these curves being concave relative to the region enclosed. In our case the three curves are segments of intersecting hyperbolae, the points of intersection marking the cusps. This is illustrated below: the steeply sloping red line is tangent to the left-most hyperbola. The pale grey lines are also tangent to the deltoid. Indeed, three of them are simultaneously tangent to two of the three curves.

The boxed image is a detail of the plot on the right. This shows a triangle with vertices $A = [2, 2]$, $B = [1, 9]$, and $C = [10, 1]$. Its edges have been extended to be more easily visualised as asymptotes to three hyperbolae. These hyperbolae intersect at points of tangency on the grey lines which join triangle vertices to the midpoints of their opposite sides. These lines are thus tangent to the deltoid, and they indeed bisect the triangle area, because they divide triangle ABC into two subtriangles of equal base and height. The hyperbolae are also tangent to the three grey lines which are parallel to the triangle edges and which have been plotted to divide the three altitudes in the ratio $1/\sqrt{2} : 1 - 1/\sqrt{2}$. Such a line also bisects the triangle area: it forms a subtriangle whose base and height are $1/\sqrt{2}$ times the base and height of triangle ABC, and therefore has half its area.





The red line is an arbitrary tangent to the ABC deltoid and therefore necessarily also bisects the triangle's area.

The content of this theorem dates in various forms at least as far back as the late 19th century. It was rediscovered, as it is presented here, in an elegant 1972 article of James Dunn and James Pretty. Many further properties of their deltoid have subsequently been discovered, for instance its area is a constant multiple of the area of the triangle, the constant being $3/4 \ln 2 - 1/2 = 0.019860\ldots$.

Web link: www.cut-the-knot.org/Curriculum/Geometry/GeoGebra/HalvedArea.shtml

Further reading: *Geometry for College Students* by I Martin Isaacs, American Mathematical Society, 2001.