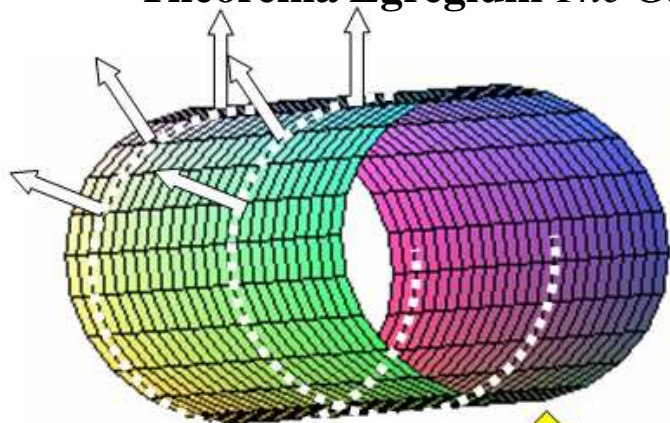


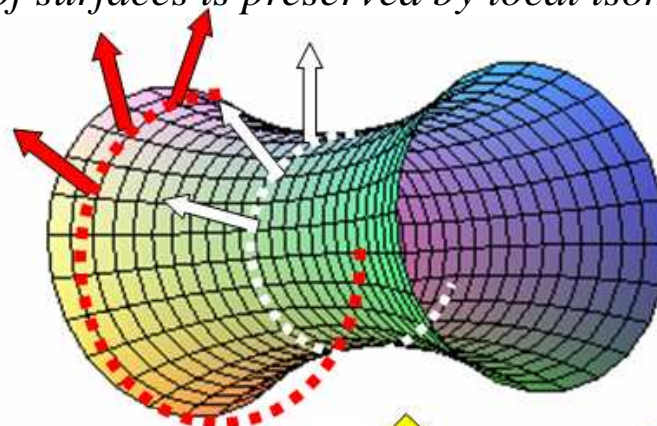


THEOREM OF THE DAY

Theorema Egregium *The Gaussian curvature of surfaces is preserved by local isometries.*

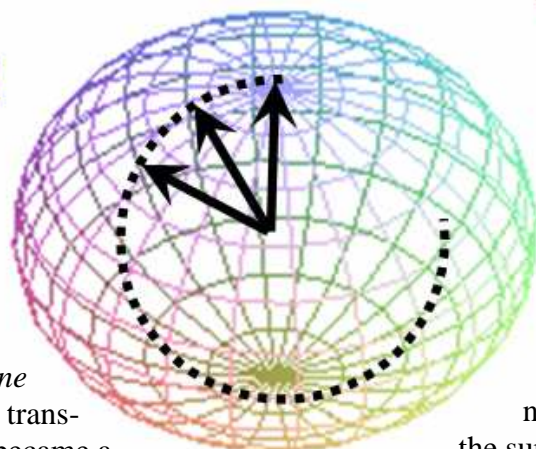


Cylinder
 $(u, \cos v, \sin v),$
 $-1 \leq u \leq 1,$
 $-\tau/2 \leq v < \tau/2$
 $(\tau = 2\pi)$

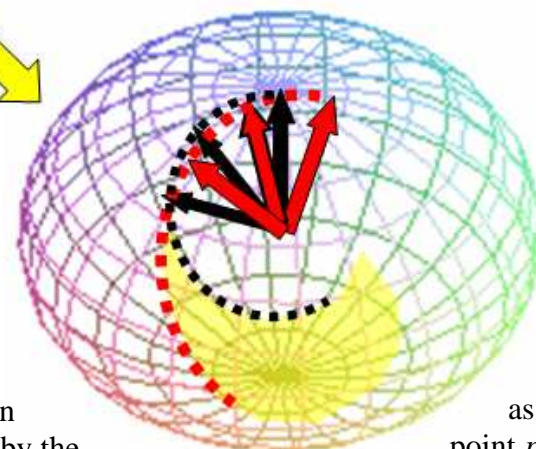


Catenoid
 $(u, \cosh u \cos v, \cosh u \sin v),$
 $-1 \leq u \leq 1,$
 $-\tau/2 \leq v < \tau/2$

Gauss discovered a wonderful way to specify how ‘curved’ a surface is: for a curve γ in 3-space we measure the rate of change of direction of the tangent vector to γ per unit length; for a surface S we measure the rate of change of the normal vector to the tangent plane per unit area. Gauss made this idea precise by translating and scaling the normal vector so that it became a radius vector in the unit sphere: this is the so-called *Gauss map* \mathcal{G} .



Now, take a small region R on S , surrounding point p . Let $\mathcal{G}(R)$ denote the region plotted on the surface of the unit sphere by the



Then $\frac{\text{area}(\mathcal{G}(R))}{\text{area}(R)}$, as R shrinks to point p , defines the Gaussian curvature K of S at p .

We can at once see that the cylinder, top-left in the illustration, has Gaussian curvature $K = 0$ everywhere. This is because all normal vectors on the surface will map to the same curve on the surface of the unit sphere (bottom-centre in the illustration); this curve has zero area, so the numerator in the definition of K vanishes. One might protest that the cylinder has a self-evidently curved surface, but Gauss was capturing a deeper distinction between surfaces. There is *no* distinction between the cylinder and a flat piece of paper in that the latter can be wrapped around the former without distorting distances; this ‘wrapping’ function is an example of a local isometry: a function which maps any curve onto one of the same length.

The value of K for the catenoid (top-right) is less immediate, but differential geometry brings calculus to our aid: we calculate K as $(LN - M^2)/(EG - F^2)$ where $E, F, G,$ and L, M and N are the coefficients of the first and second fundamental forms of the surface, respectively, calculated from its parameterisation $\sigma(u, v)$ in terms of the first and second partial derivatives with respect to u and v . For the catenoid, we derive the value $K = -\text{sech}^4 u$, a bell-shaped curve whose value is always negative.

The ‘egregious’ (remarkable) implication of Gauss’s 1827 theorem is that Gaussian curvature is an ‘intrinsic’ property of a surface. To paraphrase: a 2-D bug can determine for itself whether it lives on a cylinder or a catenoid.

Web link: www.wired.com/2014/09/curvature-and-strength-empzeal/

Further reading: *Elementary Differential Geometry* by Andrew N. Pressley, Springer, London, 2nd edition, 2010, chapters 7 and 8.

