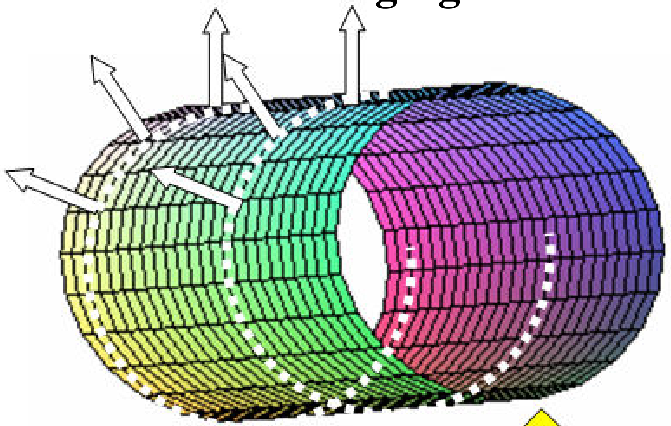
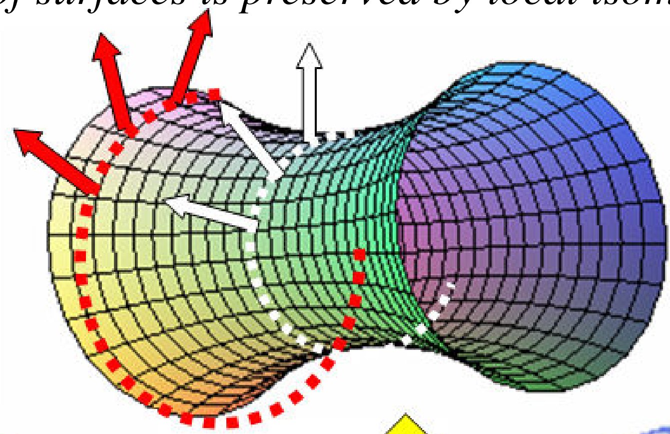
THEOREM OF THE DAY

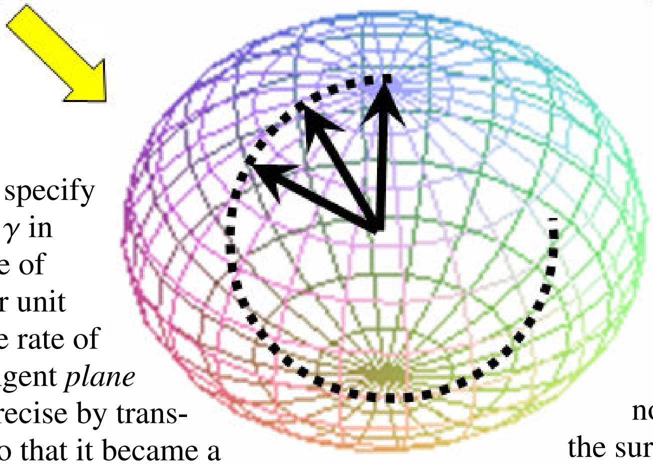
Theorema Egregium *The Gaussian curvature of surfaces is preserved by local isometries.*

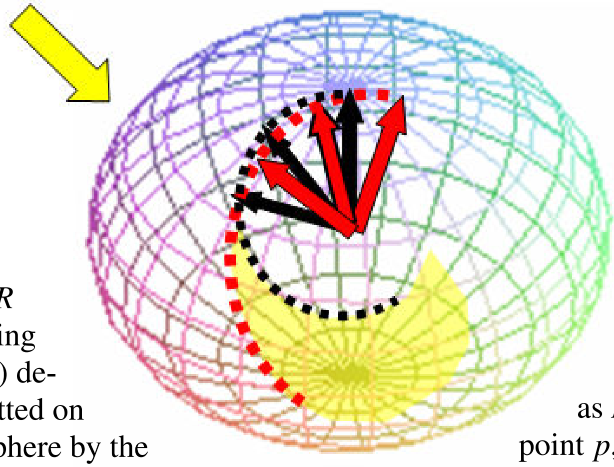


Cylinder
$(u, \cos v, \sin v)$,
$-1 \le u \le 1$,
$-\tau/2 \le v < \tau/2$
$(\tau = 2\pi)$



Catenoid
$(u, \cosh u \cos v, \cosh u \sin v)$,
$-1 \le u \le 1$,
$-\tau/2 \le v < \tau/2$





Gauss discovered a wonderful way to specify how 'curved' a surface is: for a curve γ in 3-space we measure the rate of change of direction of the tangent vector to γ per unit length; for a surface S we measure the rate of change of the normal vector to the tangent *plane* per unit *area*. Gauss made this idea precise by translating and scaling the normal vector so that it became a radius vector in the unit sphere: this is the so-called *Gauss map* $\mathcal{G}$. $\longrightarrow$

Now, take a small region R on S, surrounding point p. Let $\mathcal{G}(R)$ denote the region plotted on the surface of the unit sphere by the Gauss map of the normal vectors within R. $\longrightarrow$ $\longrightarrow$

Then $\dfrac{\text{area}(\mathcal{G}(R))}{\text{area}(R)}$, as R shrinks to point p, defines the Gaussian curvature K of S at p.

We can at once see that the cylinder, top-left in the illustration, has Gaussian curvature $K = 0$ everywhere. This is because all normal vectors on the surface will map to the same curve on the surface of the unit sphere (bottom-centre in the illustration); this curve has zero area, so the numerator in the definition of K vanishes. One might protest that the cylinder has a self-evidently curved surface, but Gauss was capturing a deeper distinction between surfaces. There is *no* distinction between the cylinder and a flat piece of paper in that the latter can be wrapped around the former without distorting distances; this 'wrapping' function is an example of a local isometry: a function which maps any curve onto one of the same length.

The value of K for the catenoid (top-right) is less immediate, but differential geometry brings calculus to our aid: we calculate K as $(LN - M^2)/(EG - F^2)$ where E, F, G, and L, M and N are the coefficients of the first and second fundamental forms of the surface, respectively, calculated from its parameterisation $\sigma(u, v)$ in terms of the first and second partial derivatives with respect to u and v. For the catenoid, we derive the value $K = -\text{sech}^4 u$, a bell-shaped curve whose value is always negative.

The 'egregious' (remarkable) implication of Gauss's 1827 theorem is that Gaussian curvature is an 'intrinsic' property of a surface. To paraphrase: a 2-D bug can determine for itself whether it lives on a cylinder or a catenoid.

Web link: people.maths.ox.ac.uk/hitchin/hitchinnotes/hitchinnotes.html (Ch. 3 of Geometry of Surfaces)
Further reading: *Elementary Differential Geometry* by Andrew N. Pressley, Springer, London, 2nd edition, 2010, chapters 7 and 8.