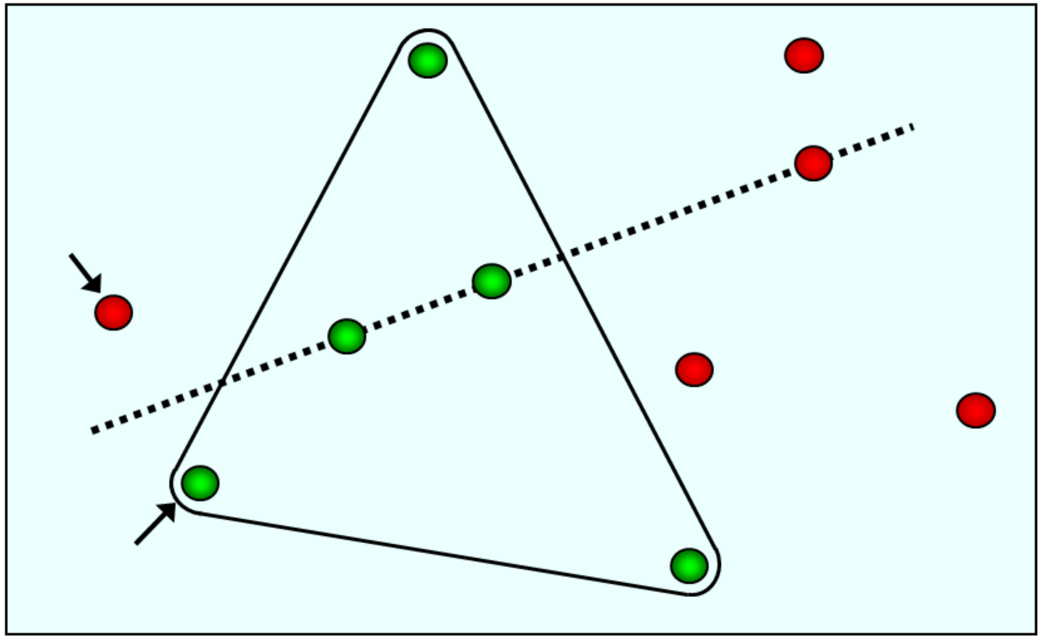
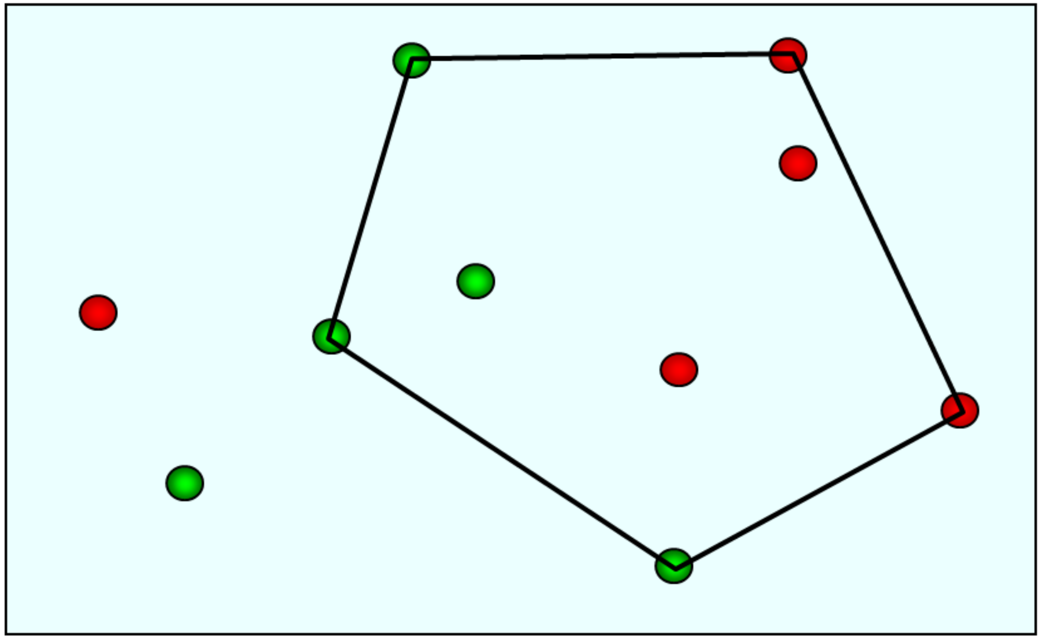
THEOREM OF THE DAY

The Happy Ending Problem *Among any five points in general position in the plane there are four which form the vertices of a convex quadrilateral.*





In n-dimensional space, a set of points is in 'general position' if no subset of k points, $2 \le k \le n+1$, is confined to a $(k-2)$-dimensional subspace. In the plane, this means that no two points should coincide (a zero-dimensional subspace), and no three should lie on a line (1-dimensional). In the collection of 10 points, above left, the dotted line shows that only 9 may be selected in general position. Any two points on this line together with the two points indicated by arrows will form an indented *non-convex* quadrilateral: some straight lines joining points within it will pass outside its boundary. So five points in general position are necessary for the theorem to be true.

Five points are here shown as being enclosed by a triangular boundary: it represents the so-called *convex hull* of the five points within it (imagine an elastic band being stretched around nails on a peg-board). When the convex hull of five points is a quadrilateral or a pentagon it is trivial to exhibit the convex quadrilateral required by the Happy Ending Problem. When it is a triangle, as above, the line joining the two internal points (our dotted line serves again) bisects the triangle; two more points are located on one side of the line or the other; the four give our quadrilateral.

How many points are required to find a pentagon? The same nine points are easily shown, above right, to supply one. It is harder to find eight points which do *not* contain a pentagon (an answer is shown here: en.wikipedia.org/wiki/Happy_Ending_problem).

Esther Klein presented this result, in 1933, by way of a problem: find an extension to n-gons, for $n > 4$. Paul Erdős and George Szekeres immediately proved that $N(n)$ exists such that $N(n)$ points in general position contain a convex n-gon; nearly 30 years later they proved that $2^{n-2}+1 \le N(n) \le \binom{2n-4}{n-2}$. The lower bound is believed to be the exact value of $N(n)$: this was proved for $n = 5$ by Endre Makai and Paul Turán, and for $n = 6$ by Szekeres and Lindsay Peters in 2006, but is open for $n > 6$. The "Happy Ending Problem" was given its name by Erdős to celebrate the happy marriage, in 1937, of Klein and Szekeres.

Web link: www.ams.org/bull/2000-37-04/: see Morris and Soltan; the $2^{n-2}+1$ bound proved asymptotically: gilkalai.wordpress.com/2016/05/04/.

Further reading: *My Brain is Open: The Mathematical Journeys of Paul Erdős* by Bruce Schechter, Prentice Hall, 2000.