THEOREM OF THE DAY

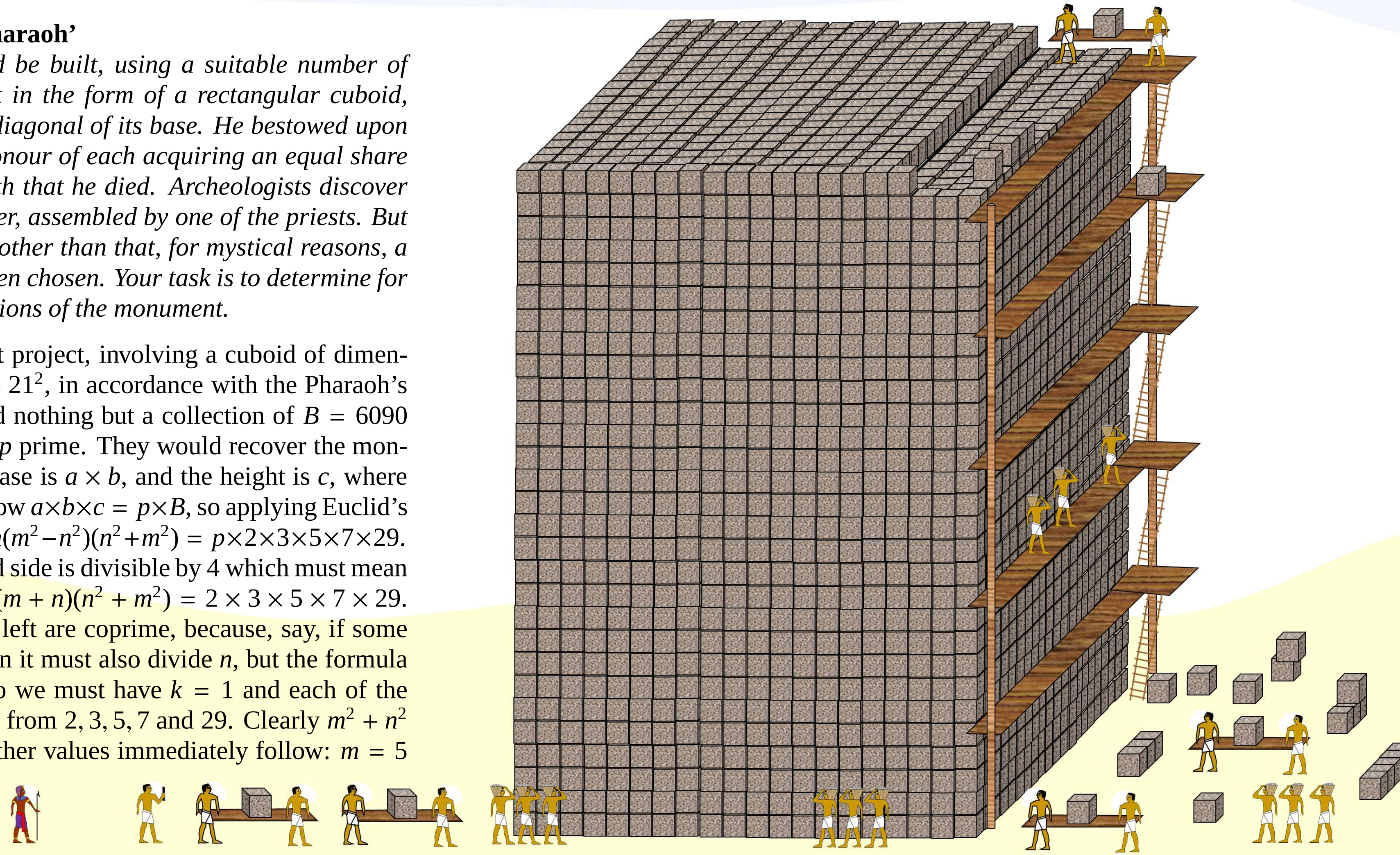
Euclid's Pythagorean Formula *For each Pythagorean triple (a, b, c) (i.e. positive integers satisfying $a^2 + b^2 = c^2$) there is a unique triple (k, m, n) of positive integers, with $m > n$, and m and n coprime and of different parity, such that*

$$a = k(m^2 - n^2),\ b = 2kmn,\ c = k(m^2 + n^2).$$

Saint-Exupéry's 'Problem of the Pharaoh'

A pharaoh decreed that there should be built, using a suitable number of 1m³ stone blocks, a giant monument in the form of a rectangular cuboid, whose height should be equal to the diagonal of its base. He bestowed upon a certain number of his priests the honour of each acquiring an equal share of the stone blocks required. And with that he died. Archeologists discover the stone blocks, 348960150 in number, assembled by one of the priests. But of the other priests nothing is known other than that, for mystical reasons, a prime number of priests must have been chosen. Your task is to determine for the archeologists the intended dimensions of the monument.

The illustration shows a more modest project, involving a cuboid of dimensions $20 \times 21 \times 29$, with $29^2 = 20^2 + 21^2$, in accordance with the Pharaoh's scheme. Suppose archeologists found nothing but a collection of $B = 6090$ blocks belonging to one of p priests, p prime. They would recover the monument's dimensions as follows: the base is $a \times b$, and the height is c, where (a, b, c) forms a Pythagorean triple. Now $a \times b \times c = p \times B$, so applying Euclid's formula we find k, m and n with $2k^3mn(m^2 - n^2)(n^2 + m^2) = p \times 2 \times 3 \times 5 \times 7 \times 29$. One of m and n is even so the left-hand side is divisible by 4 which must mean $p = 2$. So we can write $k^3mn(m - n)(m + n)(n^2 + m^2) = 2 \times 3 \times 5 \times 7 \times 29$. Moreover, the last five factors on the left are coprime, because, say, if some prime q divides both m and $m - n$ then it must also divide n, but the formula tells us that m and n are coprime. So we must have $k = 1$ and each of the other factors equals a different choice from $2, 3, 5, 7$ and 29. Clearly $m^2 + n^2$ can only take the value 29, and the other values immediately follow: $m = 5$ and $n = 2$.



Euclid's formula appears in Book 10 of his *Elements* where it is the content of Lemma 1, used in the proof of Proposition 29. The parameter k is not part of the original formula but is introduced to allow 'imprimitive' triples, such as $(9, 12, 15) = 3 \times (3, 4, 5)$, to be generated.

Web link: digitalcommons.ursinus.edu/triumphs_number/7/

Further reading: *A Pythagorean Introduction to Number Theory,* by Ramin Takloo-Bighash, Springer, 2018.