



THEOREM OF THE DAY

Euler's Formula For any real or complex value of θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Facsimile of *Introductio in analysin infinitorum* from: gallica.bnf.fr/ark:/12148/bpt6k69587

138. Ponatur denuo in formulis § 133 arcus z infinite parvus et sit n numerus infinite magnus i , ut iz obtineat valorem finitum v . Erit ergo $nz = v$ et $z = \frac{v}{i}$, unde $\sin. z = \frac{v}{i}$ et $\cos. z = 1$; his substitutis fit

$$\cos. v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2}$$

atque

$$\sin. v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}$$

In capite autem praecedente vidimus esse

$$\left(1 + \frac{z}{i}\right)^i = e^z$$

denotante e basin logarithmorum hyperbolicorum; scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit

$$\cos. v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

et

$$\sin. v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$$

Ex quibus intelligitur, quomodo quantitates exponentiales imaginariae ad sinus et cosinus arcuum realium reducuntur.¹⁾ Erit vero ...

a real page-turner from Euler: his climax to section 138 lies just overleaf from the derivation shown above! In modern-day language this derivation, taking off from De Moivre's Theorem and invoking Euler's own limit definition of e^x , reads as follows:

$$\cos \theta = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\left(1 + \frac{i\theta}{n}\right)^n + \left(1 - \frac{i\theta}{n}\right)^n \right)$$

$$i \sin \theta = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\left(1 + \frac{i\theta}{n}\right)^n - \left(1 - \frac{i\theta}{n}\right)^n \right)$$

and $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$, so

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

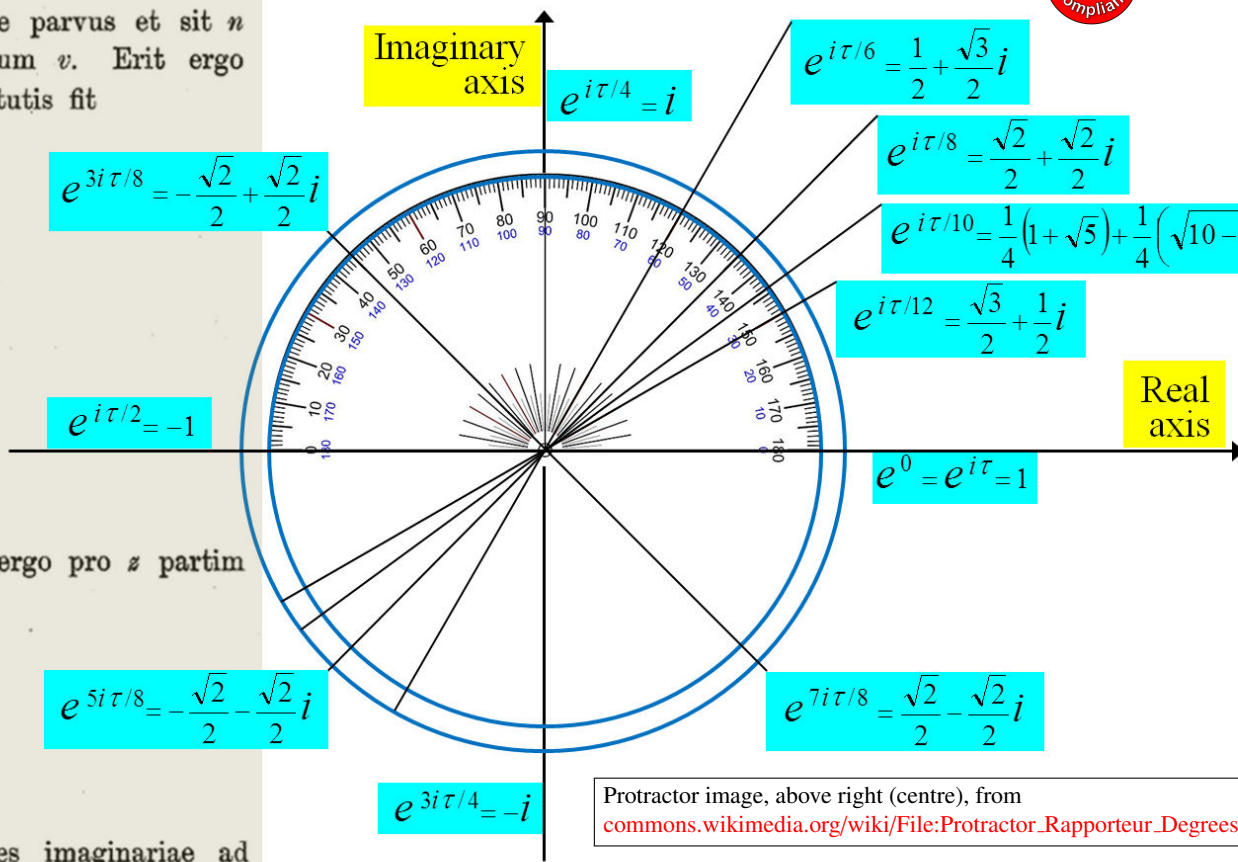
$$i \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Erit vero:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta.$$

Leonhard Euler's formula appears in vol. 1 of his 1748 *Introductio in analysin infinitorum*. It was known in the form $\ln(\cos \theta + i \sin \theta) = i\theta$ to Roger Cotes about 35 years earlier. Neither Cotes nor Euler made the connection to circle geometry: this had to wait for the invention of the complex plane by Gauss, Wessel and Argand, around 1800.



Protractor image, above right (centre), from commons.wikimedia.org/wiki/File:Protractor_Rapporteur_Degrees_V3.jpg

Web link: Ed Sandifer's *How Euler Did It*, archived at eulerarchive.maa.org/hedi/; see [August 2007 entry](#).

Further reading: *Euler: The Master of Us All* by William Dunham, MAA, 1999.