



THEOREM OF THE DAY

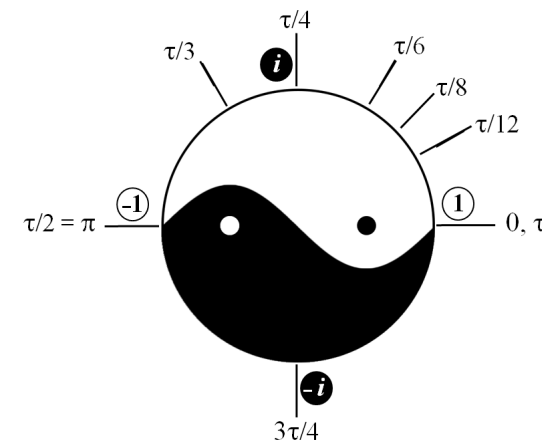
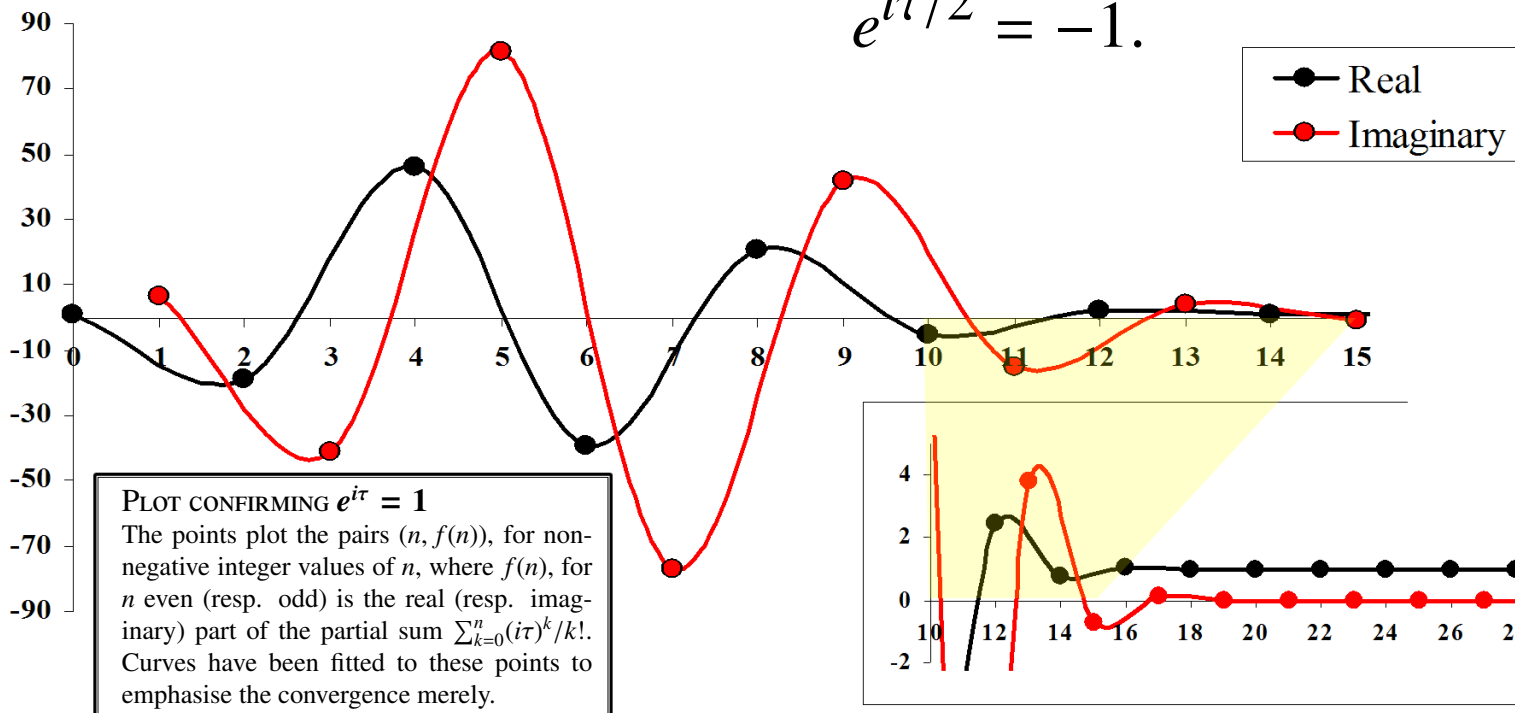
Euler's Identity With τ and e the real constants

$\tau = 6.2831853071\ 7958647692\ 5286766559\ 0057683943\ 3879875021\ 1641949889\ 1846156328\ 1257241799\ 7256069650\ 6842341359\ \dots$

and $e = 2.7182818284\ 5904523536\ 0287471352\ 6624977572\ 4709369995\ 9574966967\ 6277240766\ 3035354759\ 4571382178\ 5251664274\ \dots$

(the first 100 places of decimal being given), and i the imaginary constant satisfying $i^2 = -1$, we have

$$e^{i\tau/2} = -1.$$



Squaring both sides of $e^{i\tau/2} = -1$ gives $e^{i\tau} = 1$, encoding the defining fact that τ radians measures one full circle. The calculation can be confirmed explicitly using the evaluation of e^z , for any complex number z , as an infinite sum: $e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \dots$. Setting $z = i\tau$, the even powers of i alternate between 1 and -1 , while the odd powers alternate between i and $-i$, so we get two separate sums, one with i 's (the imaginary part) and one without (the real part). Both converge rapidly as shown in the two plots above: the real part to 1, the imaginary to 0. In the *limit* equality is attained, $e^{i\tau} = 1 + 0 \times i$, whence $e^{i\tau} = 1$. The value of $e^{i\tau/2}$ may be confirmed in the same way.

Leonhard Euler's 1748 *Introductio* presents the general circle identity $e^{i\theta} = \cos \theta + i \sin \theta$, with $\theta = \tau/2$ radians (half a turn) giving the iconic evaluation of $e^{i\tau/2}$. Although better known in the form $e^{i\pi} + 1 = 0$, $\pi = \tau/2$, the half circle angle $\tau/2$ is essential. Thus $3i\tau/2, 5i\tau/2, \dots$, also exponentiate to -1 , $\tau/2$ being distinguished as the **principal value**. A quarter turn $\tau/4$ gives $e^{i\tau/4} = i$, whence the remarkable fact that $i^i = (e^{i\tau/4})^i = e^{i^2\tau/4} = e^{-\tau/4}$, a real number. And in general $e^{i\tau/n} = \sqrt[n]{1}$, an n -th root of unity.

Web link: fermatlasttheorem.blogspot.com/2006/02/eulers-identity.html. More on i^i : www.walkingrandomly.com/?p=294.

Further reading: *Dr Euler's Fabulous Formula: Cures Many Mathematical Ills*, by Paul J. Nahin, Princeton University Press, 2006

