THEOREM OF THE DAY

Euler's Polyhedral Formula *For a connected plane graph G with n vertices, e edges and f faces,*

$$n - e + f = 2.$$

$n = 6$
$e = 7$
$f = 3$

$n = 8$
$e = 8$
$f = 4$

$n = 3$
$e = 7$
$f = 7$

Plane graphs are those which have been drawn on a plane or sphere with edges meeting only at vertices. Only for the first of the three spheres on the left does the formula appear to work ($6 - 7 + 3 = 2$) but the graph drawn on the second sphere is not connected, having three components, while that on the third has two components. We can extend Euler's formula: $n - e + f = c + 1$, where c is the number of connected components. If we take all three spheres together we get a graph with six components illustrating a further extension of the formula: $n - e + f = s + c$ where s is the number of spheres. This is one way of explaining where the number 2 in Euler's original formula comes from: one connected component drawn on one sphere.

Another explanation is found in the following
Proof: For graph G, take the dual graph G^* whose vertices are the plane faces and whose edges join dual vertices by crossing edges of G. Take a spanning tree T in G: it has n vertices and $n - 1$ edges. The remaining edges of G are crossed by a spanning tree T^* of G^*: it has f vertices and $f - 1$ edges. Every edge of G is in T or is crossed by an edge of T^*. So $e = (n - 1) + (f - 1)$. □

Even though the ancient Greeks studied convex polyhedra intensively the relation $e - n + f = 2$ appears to have been unknown before Euler spotted it in 1750. Since then it has been generalised continuously as well as finding applications in all kinds of mathematics, notably the proof of the four colour theorem. Euler did not prove the formula correctly, this being first done by Legendre in 1794; the proof given here is attributed to the geometer Karl von Staudt, 1847.

Graph G
$n = 5$
$e = 7$
$f = 4$

With dual G^*
$n = 4$
$e = 7$
$f = 5$

and spanning trees T and T^*

G
G^*
T
T^*
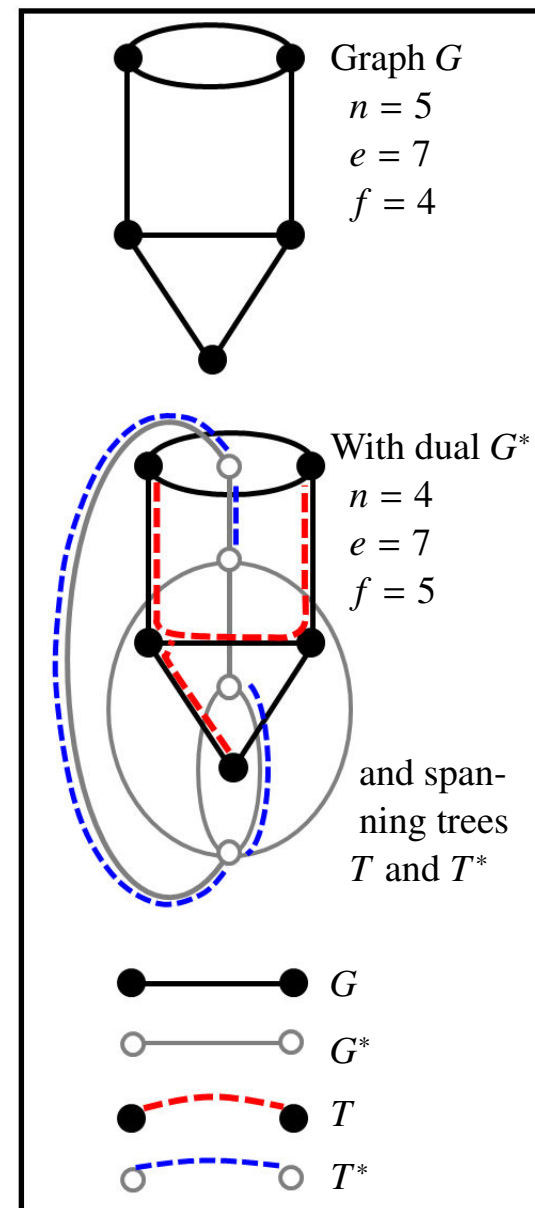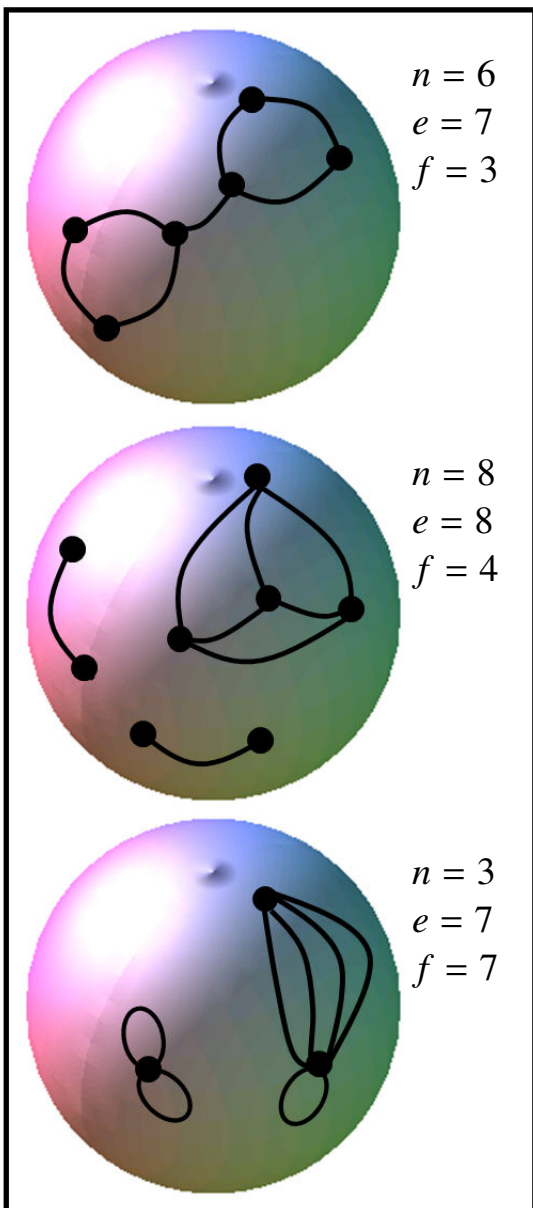
Web link: www.ics.uci.edu/~eppstein/junkyard/euler/
Further reading: *Proofs and Refutations. The Logic of Mathematical Discovery* by Imre Lakatos, edited by John Worrall and Elie Zahar, Cambridge University Press, 1976.