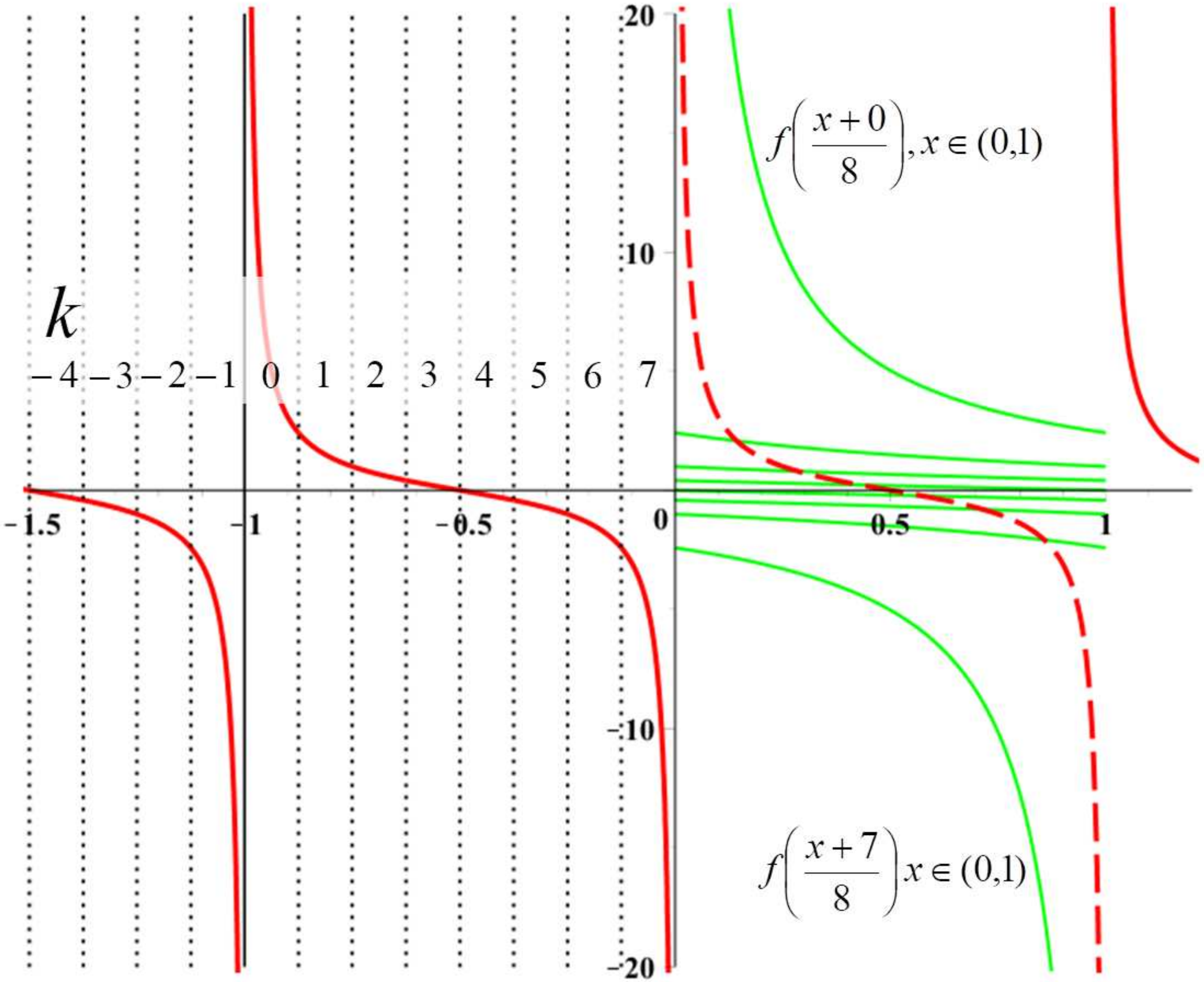
THEOREM OF THE DAY

Euler's Product Formula for Sine *For any complex number z,* $\sin\frac{\tau}{2}z = \frac{\tau}{2}z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots$.



Euler's formula, at least for real values and *glossing over several important considerations of convergence*, can be shown to emerge from an elementary but remarkable property of the cotangent function, illustrated on the left. Outside the interval [0, 1] we have plotted $f(x) = \cot(\tau x/2)$. It repeats with period 1. Suppose we divide any complete period into N equal intervals as shown here, with $N = 8$, for the negative part of the real axis. Stretch the curve in each subinterval so it fills a whole unit interval. For our illustration we plotted $f\left(\frac{x+k}{8}\right)$, for $0 \le k \le 7$, for x in (0, 1). These are the shallow green curves. Then it can be shown that these stretched curves average to give the original curve! That is, $\frac{1}{N}\sum_{k=0}^{N-1} f((x+k)/N)$, which we have plotted as the dashed curve in the interval (0, 1), is identical to $f(x)$. We need to rewrite the summands as $f((x\pm k)/N)$ and we may do this because f is periodic: our illustration shows the correspondence of positive and negative k values. Taking $N = 2^n$, we derive $f(x) = 2^{-n}\left(f(x/2^n) + f\left((x+2^{n-1})/2^n\right) + \sum_{k=1}^{2^{n-1}-1} f((x\pm k)/2^n)\right)$.
Now by L'Hospital's Rule, $\lim_{x\to 0} x\cot x = 1$ so, for non-integer α, and denoting $\tau/2$ by π, $\frac{\pi\alpha}{2^n} f\left(\frac{\alpha}{2^n}\right) \to 1$, as $n\to\infty$, or $\frac{1}{2^n} f\left(\frac{\alpha}{2^n}\right) \to \frac{1}{\pi\alpha}$. Letting n approach infinity in our expression for $f(x)$ we derive **Euler's Partial Fraction Cotangent Expansion**: *for non-integer x,* $f(x) = \frac{1}{\pi x} + \sum_{k=1}^{\infty}\frac{1}{\pi(x\pm k)}$. Rearranging gives, for $x\notin\mathbb{Z}$, $\frac{\tau}{2}\cot\frac{\tau}{2}x = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x}{x^2-k^2} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{-2x/k^2}{1-x^2/k^2}$. The latter rearrangement allows us to arrive directly at Euler's sine formula by integrating both sides: $\ln\sin\frac{\tau}{2}x = \ln x + \sum_{k=1}^{\infty}\ln\left(1-x^2/k^2\right) + \ln C = \ln\left(Cx\prod_{k=1}^{\infty}\left(1-x^2/k^2\right)\right)$. Exponentiating: $\sin\frac{\tau}{2}x = Cx\prod_{k=1}^{\infty}\left(1-x^2/k^2\right)$. And invoking L'Hospital again for $\lim_{x\to 0}\sin\frac{\tau}{2}x/x = \frac{\tau}{2}$, we find that $C = \tau/2$ and Euler's formula follows.

Euler's sine product first appears in §16 of his E41 (Eneström Index) and the cotangent expansion in E61 (again §16). Made rigorous over the complex numbers they are, respectively, applications of Weierstrass's Factorisation Theorem and Mittag-Leffler's Theorem (both 1876).

Web link: cornellmath.wordpress.com/2007/07/13/. A proof (from Andy Rich) of cot(x) averaging: theoremoftheday.org/Docs/AR0218.pdf.
Further reading: *Analysis by its History* by Ernst Hairer and Gerhard Wanner, Springer, 1996.