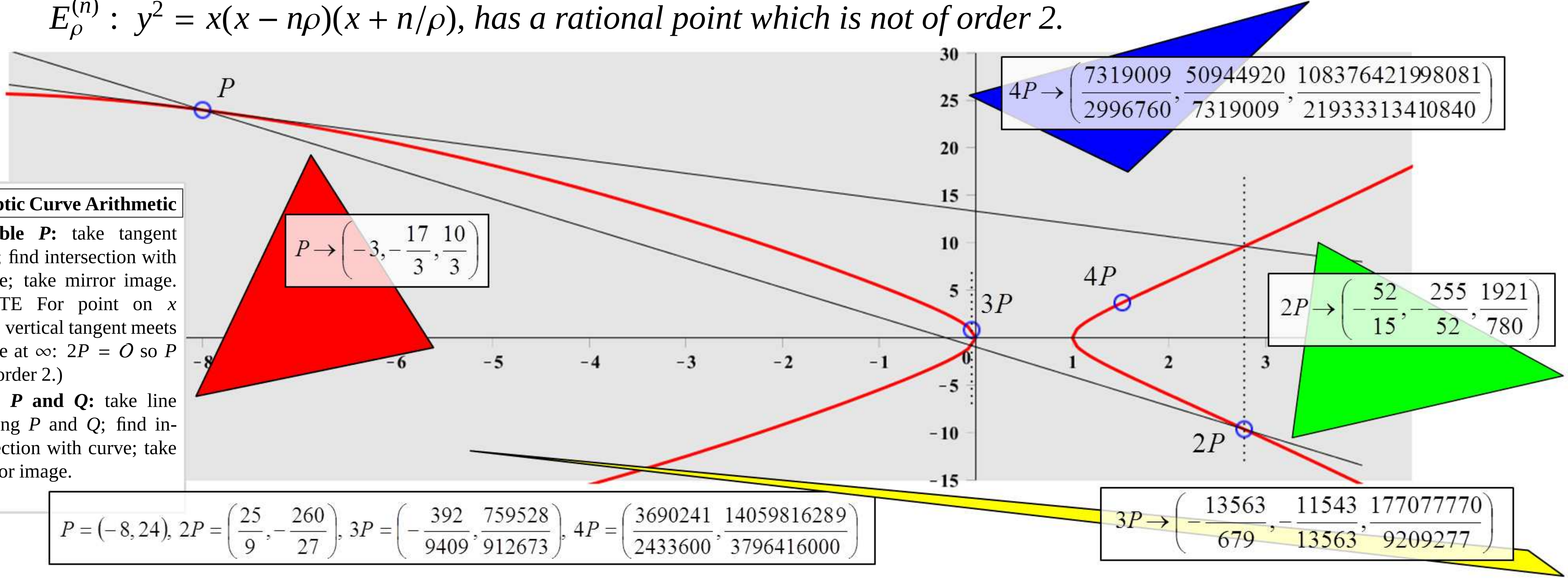
THEOREM OF THE DAY

The Goins–Maddox–Rusin Theorem on Heron Triangles *A positive integer n can be expressed as the area of a triangle with rational sides if and only if, for some nonzero, rational number ρ, the elliptic curve* $E^{(n)}_\rho\ :\ y^2 = x(x - n\rho)(x + n/\rho)$, *has a rational point which is not of order 2.*



$4P \to \left(\frac{7319009}{2996760}, \frac{50944920}{7319009}, \frac{108376421998081}{21933313410840}\right)$

$P \to \left(-3, -\frac{17}{3}, \frac{10}{3}\right)$

$2P \to \left(-\frac{52}{15}, -\frac{255}{52}, \frac{1921}{780}\right)$

$3P \to \left(-\frac{13563}{679}, -\frac{11543}{13563}, \frac{177077770}{9209277}\right)$

Elliptic Curve Arithmetic

Double P: take tangent at P; find intersection with curve; take mirror image. (NOTE For point on x axis, vertical tangent meets curve at ∞: $2P = O$ so P has order 2.)

Add P and Q: take line joining P and Q; find intersection with curve; take mirror image.

$P = (-8, 24)$, $2P = \left(\frac{25}{9}, -\frac{260}{27}\right)$, $3P = \left(-\frac{392}{9409}, \frac{759528}{912673}\right)$, $4P = \left(\frac{3690241}{2433600}, \frac{14059816289}{3796416000}\right)$

Triangles which have rational sides a, b and c and rational area n are known as *Heron triangles*; the question of their existence arises from Heron's area formula, one form of which is: $n = \frac{1}{4}\sqrt{(a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4)}$. The triangles shown above (not to scale) all have area 4 and have been produced from rational points (x, y) on the elliptic curve $E^{(4)}_{1/4}$ according to the left-hand rule below (these rules embodying a constructive proof of the theorem):

Curve, pt → △: $a = \frac{y}{x}$, $b = n\left(\rho + \frac{1}{\rho}\right)\frac{x}{y}$, $c = \frac{x^2+n^2}{y}$. **△ →Curve, pt**: $n = \frac{1}{4}\sqrt{(a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4)}$, $\rho = \frac{4n}{(a+b)^2 - c^2}$, $x = \frac{(a+c)^2 - b^2}{4}$, $y = a\frac{(a+c)^2 - b^2}{4}$.

Notice that negative ordinates may lead to triangles with negative side-lengths; Heron's formula 'ignores' signs, and the triangles have been drawn using absolute values. The proof yields much more than just the Heron triangles: they are, in addition, isosceles precisely if there are 8 rational points of finite order constituting the *torsion group* $\mathbb{Z}_2 \times \mathbb{Z}_4$. (In the example here, $y^2 = x(x-1)(x+16)$, the torsion group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, comprising the three points of order two on the x-axis and the identity point O at ∞.) The triangles are right-angled (in which case n, scaled to be integer-valued, is called a *congruent number*) precisely if ρ can be chosen to be unity.

Heron triangles and congruent numbers were studied by Fibonacci and Fermat. Modern progress dates back to an influential 1976 paper by Nathan Fine (1916–1994). The analysis described here is subsequent work of Dave Rusin, with the actual theorem as stated being due to Edray Herber Goins and Davin Maddox.

Web link: projecteuclid.org/euclid.rmjm/1181069379 (Goins–Maddox) and nyjm.albany.edu/j/1998/Vol4.html (Rusin, the first article).

Further reading: *Introduction to Elliptic Curves and Modular Forms, 2nd Edition* by Neal Koblitz, Springer-Verlag, New York, 1993.