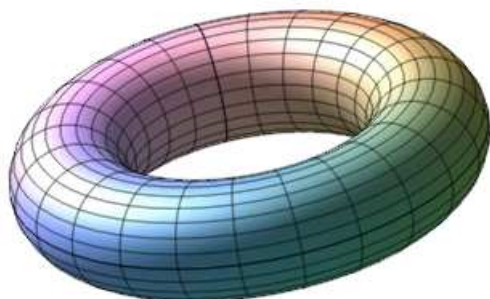




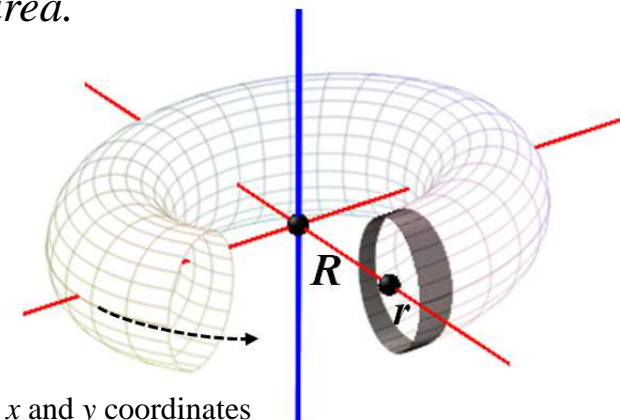
THEOREM OF THE DAY

The Pappus–Guldin Theorems Suppose that a plane curve is rotated about an axis external to the curve. Then

1. the resulting surface area of revolution is equal to the product of the length of the curve and the displacement of its centroid;
2. in the case of a closed curve, the resulting volume of revolution is equal to the product of the plane area enclosed by the curve and the displacement of the centroid of this area.



A classic example is the measurement of the surface area and volume of a torus. A torus may be specified in terms of its **minor radius** r and **major radius** R by rotating through one complete revolution (an angle of τ radians) a circle of radius r about an axis lying in the plane of the circle and at perpendicular distance R from its centre. In the image on the right, the surface and volume being generated by the rotation have area $A = \tau r \times \tau R = \tau^2 r R$ and volume $V = \frac{1}{2} \tau r^2 \times \tau R = \frac{1}{2} \tau^2 r^2 R$, respectively.



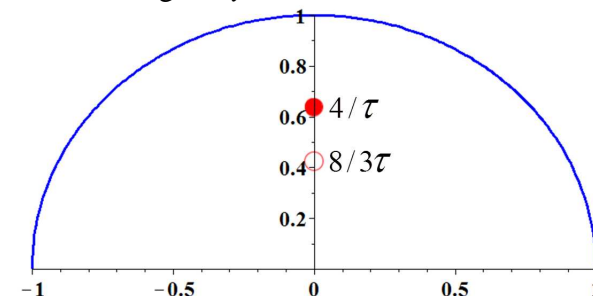
Using calculus, the centroid of the region bounded by the curve $y = f(x)$ and the x -axis in the interval $[a, b]$ has x and y coordinates

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx \quad \text{and} \quad \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 dx,$$

where A is the area of the region. Now the second Pappus–Guldin theorem gives the volume when this region is rotated through τ radians as $V = A \times \tau \bar{y} = \frac{1}{2} \tau \int_a^b (f(x))^2 dx$, the familiar formula for volume of solid of revolution. A similar calculation may be made using the y coordinate of the centroid of the arc on the curve $y = f(x)$, on the interval $[a, b]$, the coordinates of this centroid being given as:

$$\bar{x} = \frac{1}{L} \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{and} \quad \bar{y} = \frac{1}{L} \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

with L , the arc length, calculated as $\int_a^b \sqrt{1 + (dy/dx)^2} dx$. This is illustrated, right, for a semicircle: $y = \sqrt{1 - x^2}$, in the interval $[-1, 1]$. The centroid of the enclosed region is $(0, 8/3\tau)$, plotted as the outline circle; the centroid of the semicircular arc is $(0, 4/\tau)$, plotted as the solid circle.



Pappus stated his theorems in the early 300s; it is accepted that 17th century scientists rediscovered the theorems for themselves, Book II (1640) of *Centrobaryca*, Paul Guldin's 700-page treatise on centres of gravity, being the pre-eminent contribution.

Web link: old.maa.org/press/periodicals/convergence/james-gregory-and-the-pappus-guldin-theorem. Integration formulae for mensuration and location of centroids are elegantly covered in archive.uea.ac.uk/jtm/contents.htm, Section 13.

Further reading: *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* by Paolo Mancosu, Oxford University Press, 1999, chapter 2.

