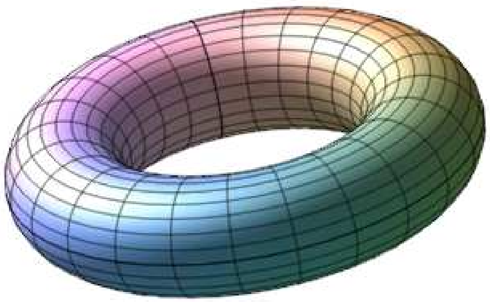
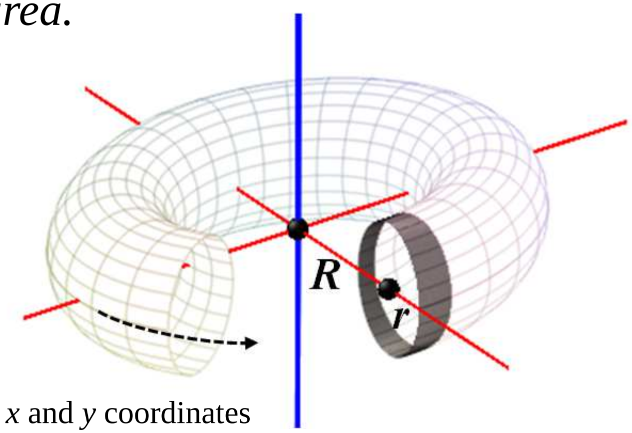
THEOREM OF THE DAY

The Pappus–Guldin Theorems *Suppose that a plane curve is rotated about an axis external to the curve. Then*

1. *the resulting surface area of revolution is equal to the product of the length of the curve and the displacement of its centroid;*
2. *in the case of a closed curve, the resulting volume of revolution is equal to the product of the plane area enclosed by the curve and the displacement of the centroid of this area.*



A classic example is the measurement of the surface area and volume of a torus. A torus may be specified in terms of its **minor radius** r and **major radius** R by rotating through one complete revolution (an angle of τ radians) a circle of radius r about an axis lying in the plane of the circle and at perpendicular distance R from its centre. In the image on the right, the surface and volume being generated by the rotation have area $A = \tau r \times \tau R = \tau^2 rR$ and volume $V = \frac{1}{2}\tau r^2 \times \tau R = \frac{1}{2}\tau^2 r^2 R$, respectively.



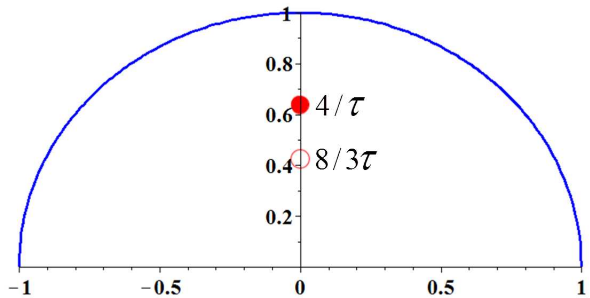
Using calculus, the centroid of the region bounded by the curve $y = f(x)$ and the x-axis in the interval $[a, b]$ has x and y coordinates

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\, dx \quad \text{and} \quad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\, dx,$$

where A is the area of the region. Now the second Pappus–Guldin theorem gives the volume when this region is rotated through τ radians as $V = A \times \tau\overline{y} = \frac{1}{2}\tau\int_a^b (f(x))^2\, dx$, the familiar formula for volume of solid of revolution. A similar calculation may be made using the y coordinate of the centroid of the arc on the curve $y = f(x)$, on the interval $[a, b]$, the coordinates of this centroid being given as:

$$\overline{x} = \frac{1}{L}\int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{and} \quad \overline{y} = \frac{1}{L}\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

with L, the arc length, calculated as $\int_a^b \sqrt{1 + (dy/dx)^2}\, dx$. This is illustrated, right, for a semicircle: $y = \sqrt{1 - x^2}$, in the interval $[-1, 1]$. The centroid of the enclosed region is $(0, 8/3\tau)$, plotted as the outline circle; the centroid of the semicircular arc is $(0, 4/\tau)$, plotted as the solid circle.



Pappus stated his theorems in the early 300s; it is accepted that 17th century scientists rediscovered the theorems for themselves, Book II (1640) of *Centrobaryca*, Paul Guldin's 700-page treatise on centres of gravity, being the pre-eminent contribution.

Web link: www.maa.org/publications/periodicals/convergence/james-gregory-and-the-pappus-guldin-theorem. Integration formulae for mensuration and location of centroids are elegantly covered in archive.uea.ac.uk/jtm/contents.htm, Section 13.

Further reading: *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* by Paolo Mancosu, Oxford University Press, 1999, chapter 2.