THEOREM OF THE DAY

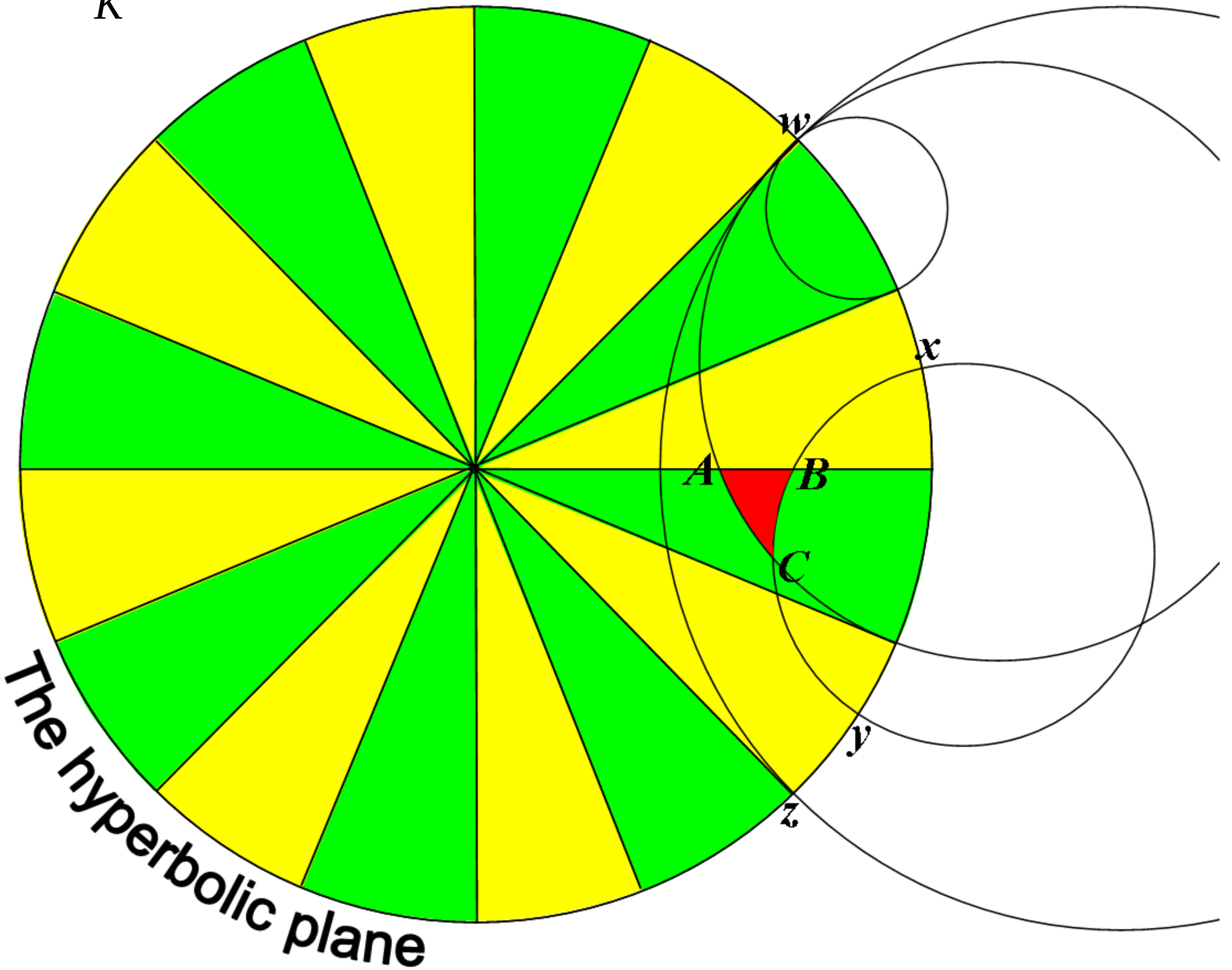
Lambert's Formula *If T is a triangle in the hyperbolic plane of (negative) Gaussian curvature K, with angles α, β and γ, then the area of T is given by*

$$\text{Area of } T = \frac{1}{K}(\alpha + \beta + \gamma - \tau/2),$$

where $\tau = 2\pi$.

The hyperbolic plane is a construction of two-dimensional geometry in which Euclid's famous *parallel postulate*, that exactly one line parallel to a given line L passes through a given point not on L, fails to hold. A circular representation of the hyperbolic plane is shown on the right in which the shortest distance joining two points, the *geodesic*, is defined by the unique circle passing through the points and intersecting the boundary of the plane at right angles. Thus, the geodesic for points B and C is given by the circle intersecting the boundary at x and y; the geodesic for points A and B, on the other hand, is given by a circle of infinite radius forming a diameter of the hyperbolic plane. In this representation, often called the Poincaré disc model but actually due to Eugenio Baltrami (1835–1900), the actual distance from B to C is calculated in terms of Euclidean distances from x and y (xC = distance x to C, etc) as $\left(1/\sqrt{-K}\right)\log(xC \times yB/xB \times yC)$. With this definition, any point in the plane is at infinite distance from the boundary. We say that two lines are parallel if they never meet, or if they meet at infinity. Thus parallel to the line wz, at least two lines pass through point C: the line wC, which meets wz at infinity, and the line xC which never meets wz.

In the hyperbolic plane, the angles of any triangle, for instance those at A, B and C here, sum to less than $\tau/2$ radians, and Lambert's formula asserts that the discrepancy is proportional to the triangular area.



Johann Heinrich Lambert (1738–1777) was one of the first to explore the properties of hyperbolic geometry. His formula is analogous to Girard's Theorem for triangular area in spherical geometry; indeed, if the 'radius' of the hyperbolic plane is defined to be the imaginary number $\sqrt{1/K}$, then Lambert's formula gives the area of a triangle on a sphere of imaginary radius.

Web link: www.theiff.org/oexhibits/oe1.html

Further reading: *The Road to Reality* by Roger Penrose, Vintage, 2005, chapter 2.