THEOREM OF THE DAY

The Power of a Point Theorem In the Euclidean plane, let $C$ be a circle of radius $r$. Let $P$ be a point whose distance from the centre of $C$ is $s$, and define the power of $P$ relative to $C$ to be the constant $h = |s^2 - r^2|$. Then for any line through $P$ intersecting $C$, $h$ is equal to the product of the distances from $P$ to its points of intersection with $C$.

**Intersecting Chords Theorem:** if $P$ is inside circle $C$, then for chords $WX$ and $YZ$ through $P$, we have $WP \cdot PX =YP \cdot PZ$. Thus, in our picture, humble little country Erewhon wishes it were part of the mighty EU; at least their flags, intersecting at $P$ and having widths $PX$ and $PZ$, respectively, have the same area...

It might not seem obvious that our points $W, X, Y$ and $Z$, regardless of the angle formed by the flagstaffs, will form a cyclic quadrilateral but in fact, as a converse result, cyclicity follows for arbitrary line segments $WX$ and $YZ$ meeting at $P$ whenever $WP \cdot PX =YP \cdot PZ$.

The idea of the power of a point was first formulated by Louis Gaultier in 1813; the name (Potenz des Puncts) first appears in Jakob Steiner’s comprehensive (and probably independent) treatment of 1826. It allows wide-reaching generalisations of various theorems in Euclidean geometry. Thus the Intersecting Chords Theorem is Proposition 35 of Book 3 of Euclid’s *Elements* (where it is asserted, as in our illustration, as an equality of areas), while the Tangent-Secant Theorem is Proposition 36.

Tangent-Secant Theorem: if $P$ is outside circle $C$, then there is a ‘double’ intersection where a tangent through $P$ touches $C$ and the distance from $P$ to either of its points of tangency is $\sqrt{h}$.

(The Pythagorean theorem follows: let $s$ and $r$ be two sides of a right triangle, with $s$ the hypotenuse, and place the centre of a circle of radius $r$ at their intersection. Then since tangents meet radii at right angles, the third side of the triangle is tangent to the circle and consequently has length $\sqrt{s^2 - r^2}$.)

**Proof of the Power of a Point Theorem:** triangles $WPZ$ and $YPX$ are similar because their angles are equal. For example, $\angle XWZ = \angle XYZ$ because angles subtending the same arc ($XZ$) are equal; and $\angle XWZ = \angle PWZ$ while $\angle XYZ = \angle YWP$. In similar triangles, ratios of corresponding sides are equal, so $WP/YP = PZ/PX$ and therefore products $WP \cdot PX$ and $YP \cdot PZ$ are equal. Now choose $WX$ to be a diameter. Then with $P$ inside the circle (and nearer to $W$ than $X$), $WP \cdot PX = (r - s)(s + r) = r^2 - s^2$. If $P$ is outside the circle, the product is $s^2 - r^2$.

Further reading: *College Geometry* by Howard Eves, Jones and Bartlett, 1995.