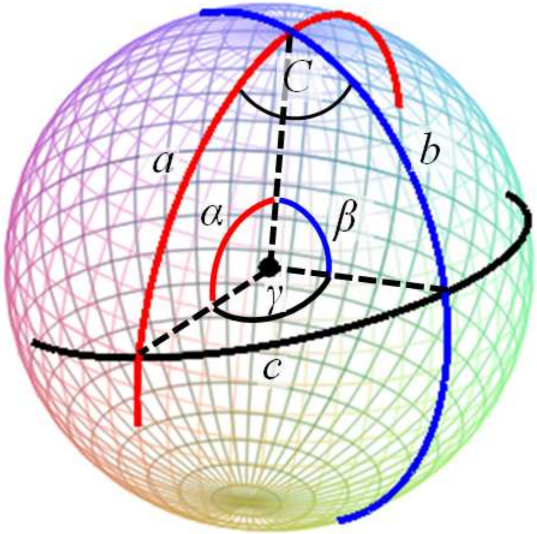
THEOREM OF THE DAY

The Spherical Law of Cosines *Suppose that a spherical triangle on the unit sphere has side lengths a, b and c, and let C denote the angle adjacent to sides a and b. Then (using radian measure):*

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C).$$

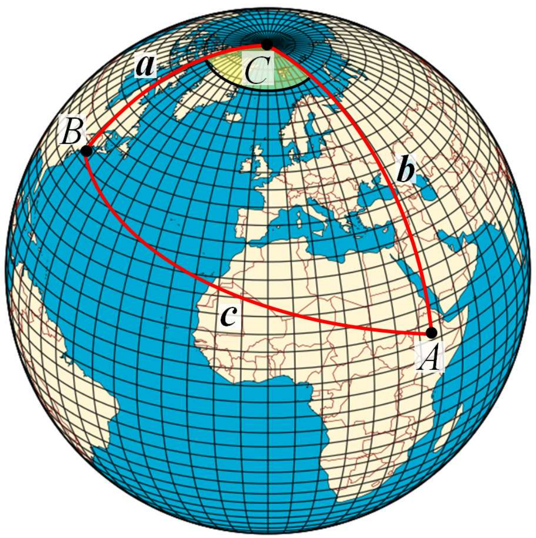


A spherical triangle is one enclosed by three great circles (each having radius 1 and common centre with the unit sphere). In the illustration on the left, three great circles are shown (each for the sake of clarity plotted through one half turn—$\tau/2$ radians) forming the triangle *abc* of the theorem. It may seem curious to take sines and cosines of side-lengths but in fact these are simultaneously the *central angles*, shown here as α, β and γ, which subtend sides a, b and c, respectively, at the centre. In fact, more generally, we may take a sphere of radius r and replace $\cos(a)$ with $\cos(a/r)$, $\cos(b)$ with $\cos(b/r)$, etc, because a/r will be the angle subtending a great circle arc of length a, and so on.

In the case where angle C is $\tau/4$ radians, so that $\cos(C) = 0$, our theorem reduces to the **Spherical Pythagorean Theorem:** $\cos(c) = \cos(a)\cos(b)$. Moving to a sphere of radius r this becomes $\cos(c/r) = \cos(a/r)\cos(b/r)$, and applying the series expansion for cosine we have

$$1 - \frac{1}{2!}\left(\frac{c}{r}\right)^2 + \frac{1}{4!}\left(\frac{c}{r}\right)^4 - \ldots = \left(1 - \frac{1}{2!}\left(\frac{a}{r}\right)^2 + \frac{1}{4!}\left(\frac{a}{r}\right)^4 - \ldots\right)\left(1 - \frac{1}{2!}\left(\frac{b}{r}\right)^2 + \frac{1}{4!}\left(\frac{b}{r}\right)^4 - \ldots\right).$$

If we multiply through by r^2 and rearrange terms then, letting r tend to infinity, we have, in the limit, $c^2 = a^2 + b^2$, for right triangles on the sphere of infinite radius, i.e. the Euclidean plane (the background image illustrates the idea, below right, of an 'almost planar' spherical triangle). The Euclidean Law of Cosines may be derived from its spherical counterpart in the same way (but note that most proofs of the latter use the former, either directly or invoked via vector algebra, so that this derivation risks circularity).



Globe image: commons.wikimedia.org/wiki/User:Hellerick

Central angles are the basis of latitude measurement. On the globe on the left, Boston (B) and Addis Ababa (A) are plotted as the vertices of a spherical triangle with apex at the North Pole (C). Boston is at latitude $42°19'N$ which means edge a is subtended by a central angle of $90° - 42°19' \approx 47.68°$. Similarly Addis Ababa, latitude $9°03'N$, gives the angle subtending edge b as about $80.95°$. And angle C is the sum of the longitudes: $71°05'W + 38°42'E \approx 109.78°$. So the great circle distance c from B to A is given approximately by

$$\cos^{-1}(\cos(48°)\cos(81°) + \sin(48°)\sin(81°)\cos(110°))$$

which is about $98° = 98 \times 60$ nautical miles (1 nautical mile = 1 minute of great circle arc), or about 10900km.

Spherical triangles have been studied since antiquity because of their importance in navigation and astronomy, but the Spherical Law of Cosines is first found in the work of Regiomontanus (1464).

Web link: www.math.sunysb.edu/~tony/archive/hon101s08/spher-trig.html for history and applications to spherical navigation

Further reading: *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry* by Glen Van Brummelen, Princeton University Press, 2012, chapter 6.