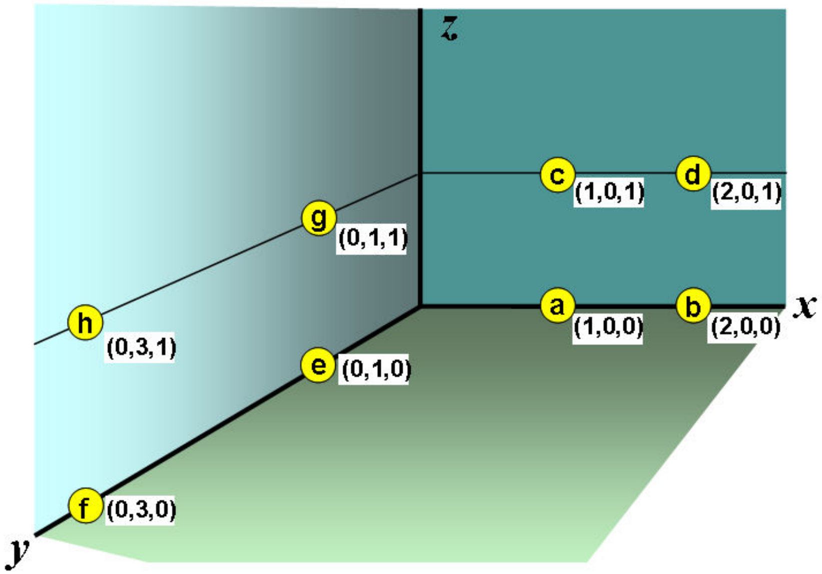
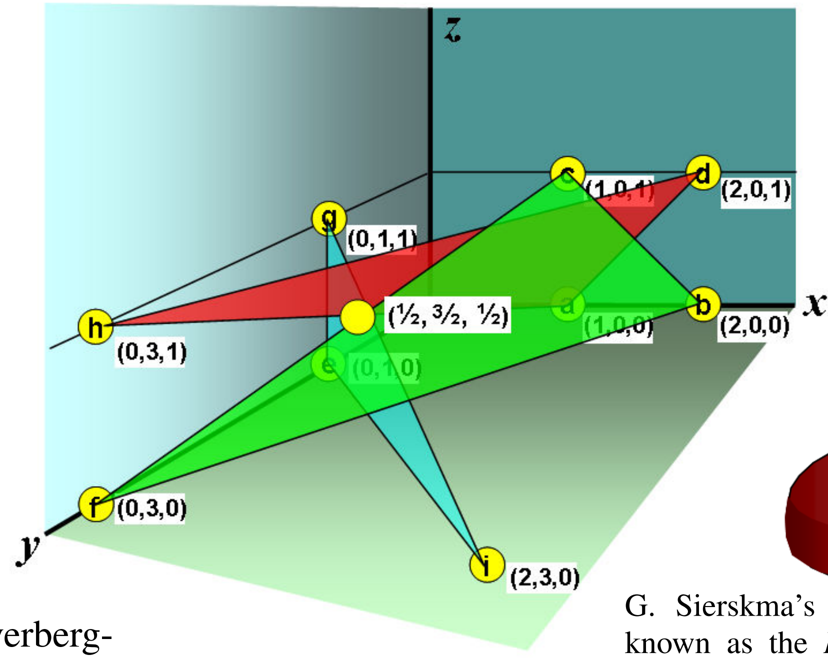
THEOREM OF THE DAY

The Tverberg Partition Theorem *A set S of points in d-dimensional Euclidean space, whose cardinality exceeds $(d+1)(r-1)$, r a positive integer, can be partitioned into r subsets, $S_1, S_2, \ldots, S_r$, whose convex hulls have non-empty intersection:*

$$S = \bigcup_{i=1}^{r} S_i, \qquad \text{for } i \neq j, S_i \cap S_j = \emptyset, \qquad \bigcap_{i=1}^{r} \text{conv}(S_i) \neq \emptyset.$$

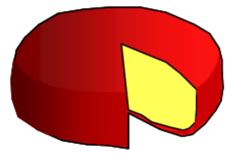


With $d = r = 3$, Tverberg's theorem requires at least $1 + (3-1)(3+1) = 9$ points. **Left:** a set of 8 points which cannot be partitioned into three subsets satisfying the conclusion of Tverberg's theorem. **Below:** by adding any single distinct point we can apply Tverberg's theorem. Here the point $i = (2,3,0)$ has been added; the 9 points have been partitioned into 3 subsets of 3 points (whose convex hulls are triangles), intersecting in the point $(1/2, 3/2, 1/2)$.



There is an important 1979 conjecture of Gerard Sierksma which says that there are at least $((r-1)!)^d$ partitions satisfying Tverberg's theorem. There are a total of six different 3-triangle partitions of the nine integer points on the right; as an exercise you may like to try to find a further two partitions to reach Sierksma's lower bound of: $((3-1)!)^3 = 8$.

In 1966 the Norwegian Helge Tverberg published this extension, to arbitrary r, of the 1921 theorem of Johann Radon (1887–1956) which asserted the case $r = 2$. The further study of Tverberg-type partitions has established itself as a flourishing branch of combinatorial geometry.



G. Sierskma's conjecture is known as the *Dutch Cheese Problem* because he offered a Dutch cheese for its solution!

Web link: Gunter M. Ziegler's Communication at www.ams.org/notices/201104/

Further reading: *Lectures in Discrete Geometry* by Jiří Matoušek, Springer-Verlag, New York, 2002 (chapter 8).