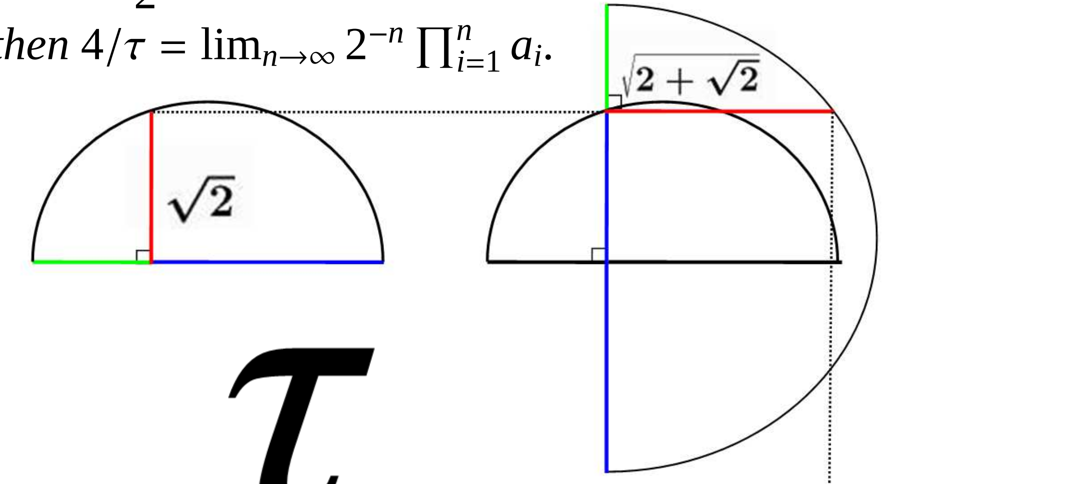
THEOREM OF THE DAY

Viète's Formula *The value of τ, the circumference of the unit circle, satisfies*

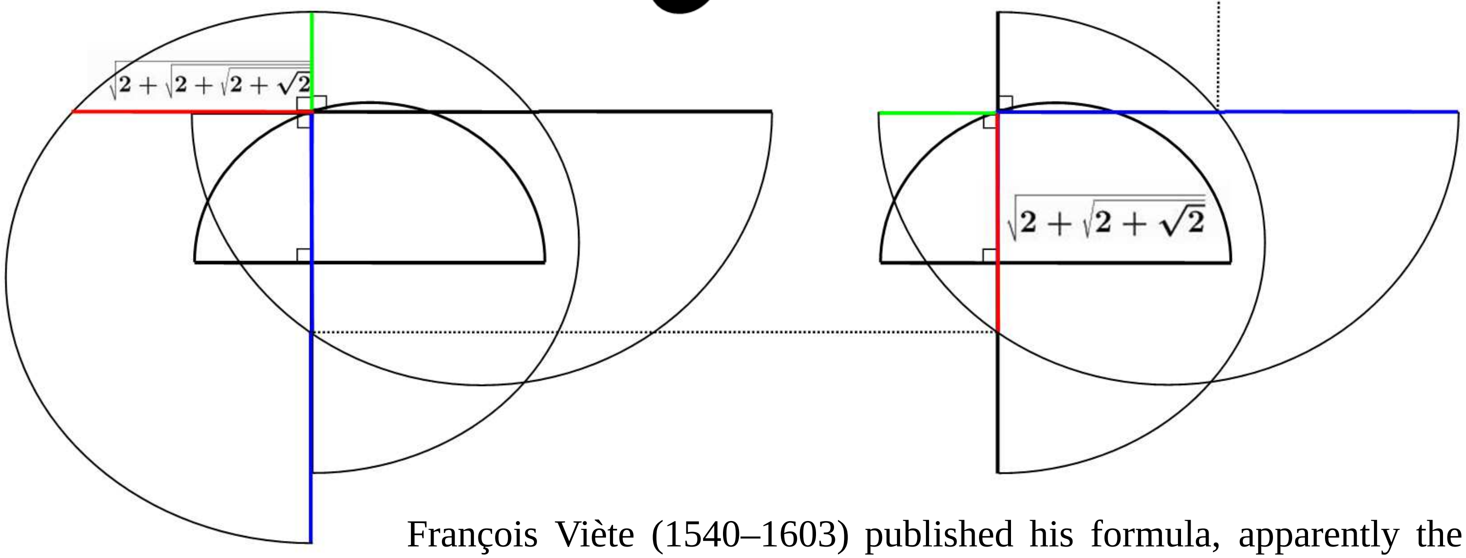
$$\frac{4}{\tau} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots,$$

that is to say, if $a_0 = 0$ *and* $a_{i+1} = \sqrt{(2+a_i)}$, $i \geq 0$, *then* $4/\tau = \lim_{n\to\infty} 2^{-n}\prod_{i=1}^{n} a_i$.



Calculating the value of τ is, in one sense, equivalent to the ancient problem of squaring the circle: for a given circle, construct, using a straightedge and compass, a square of equal area. It was finally established in the nineteenth century that only values which were built up from 'nested' square roots were 'constructible'. The construction of the nested square roots needed in Viète's Formula is illustrated opposite, starting, top left, with $\sqrt{2}$: append to a length of 2 a unit length (considered as a given). Make the total length the diameter of a circle and erect to the circle from the 1-unit point a perpendicular. This is seen to have length $\sqrt{2}$. The other diagrams repeat this process to take square roots of square roots.

Now τ cannot, in fact, be constructed by a finite number of ruler and compass operations, but Viète's formula shows that it may be approximated arbitrarily closely in this way: the more times we iterate the above process, the more decimal digits of τ we will produce. The formula may be rearranged in various ways. For instance, we can invert the left-hand side, bringing τ to the top, while hardly disturbing the right-hand side: with the a_i as in the theorem, let $b_i = 2 - a_i$, $i \geq 0$. Then $\tau/4 = \lim_{n\to\infty} 2^{n-2} b_{n-1}\prod_{i=1}^{n} a_i$. (The value of b_i goes to zero rapidly, however, making this an inaccurate way to calculate τ; and indeed Viète's formula is not of practical interest in this respect.)



François Viète (1540–1603) published his formula, apparently the earliest known infinite product approximation of a number, in 1593.

Web link: Samuel G. Moreno & Esther M. García-Caballero at www.sciencedirect.com/journal/journal-of-approximation-theory/vol/174/
Further reading: *Pi: a Source Book, 3rd edition*, by J. Lennart Berggren, J. M. Borwein and P. B. Borwein, Springer-Verlag, New York, 2004.