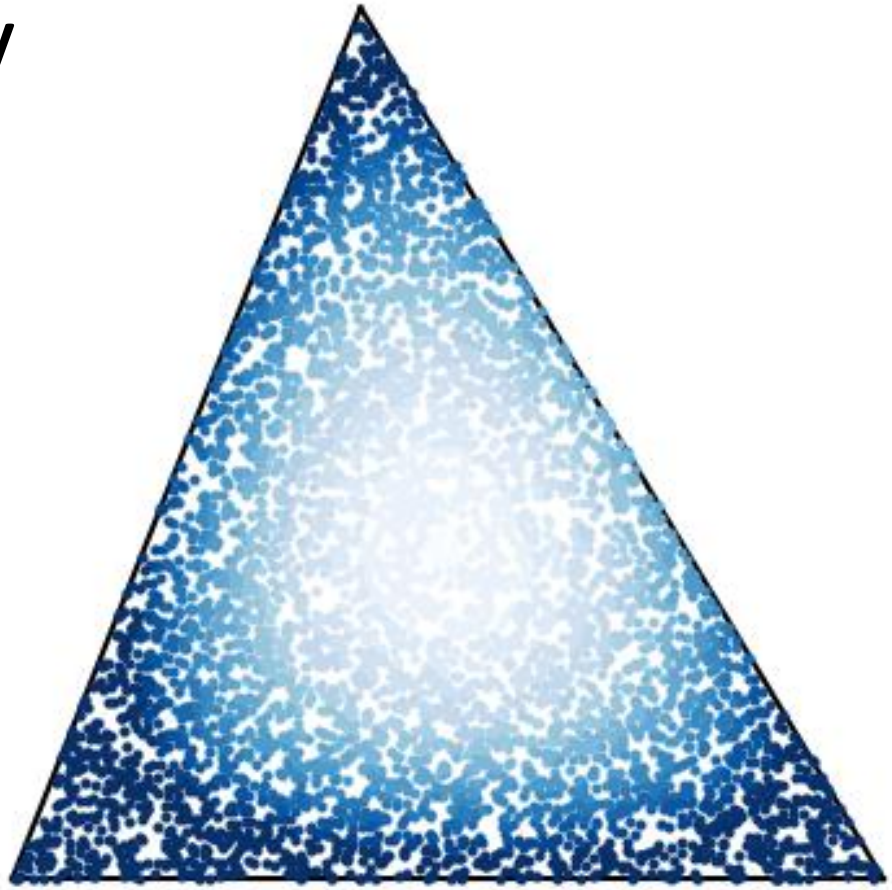
The Erdős–Mordell–Barrow Inequality



Robin Whitty, LSBU Maths Study Group,
December 2025

A problem of Paul Erdős

PROBLEMS FOR SOLUTION

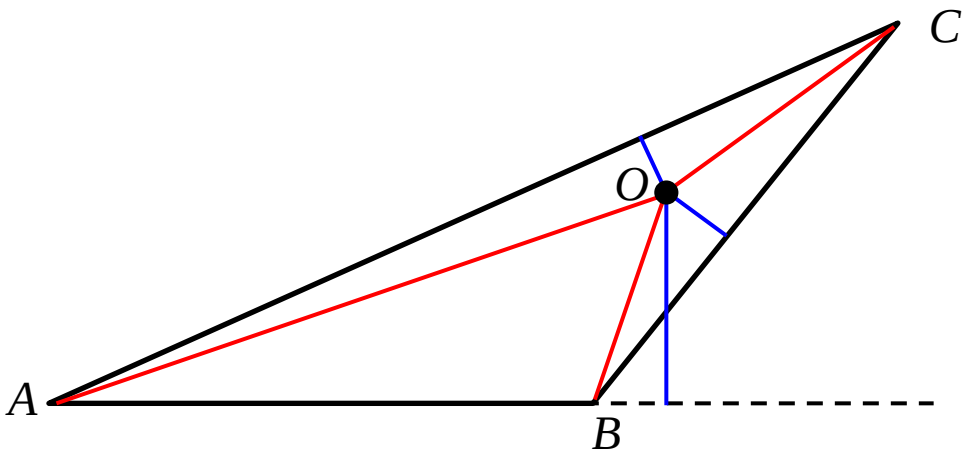
3739. *Proposed by Paul Erdös, The University, Manchester, England.*

Given $n+1$ integers, $a_1, a_2, \cdots, a_{n+1}$, each less than or equal to $2n$, prove that at least one of them is divisible by some other of the set.

3740. *Proposed by Paul Erdös, The University, Manchester, England.*

From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$



The American Mathematical Monthly, Vol. 42, No. 6 (Jun. - Jul., 1935), pp. 396–397

Paul Erdős in Manchester 1934 – 1938

"Mordell says that P Erdős, a young Hungarian, is very anxious to go to Princeton for the session 1936-7. He is undeniably a very remarkable man, extraordinarily quick and fertile; …

His line is theory of numbers in the main; and I am out of touch with it. But I have read some of his papers, and they have an undeniable touch of quality."

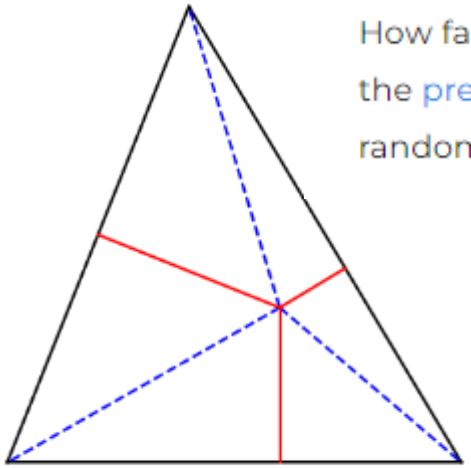
GH Hardy to Oswald Veblen, Nov. 1935. In the end Erdős went to Princeton late in 1938.

Quote from: mathshistory.st-andrews.ac.uk/Biographies/Erdos/

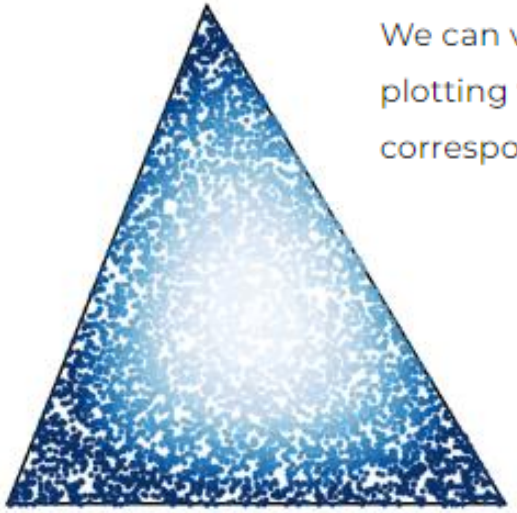
Erdős with the number theory group at the University of Manchester in 1937 or 1938. Erdős can be identified as the only one not wearing a necktie. Chao Ko stands at far right, next to Guy Davenport. Louis Mordell, who invited Erdős to come to Manchester, is the second figure to the left of Erdős.

(Reproduced with permission from the Center for Excellence in Mathematical Education, Colorado Springs)

Image: manling.wordpress.com/2008/09/30/erdos-kudos/



How far apart are the left and right sides of the inequality? This was the motivation for the previous post on selecting random points from a triangle. I wanted to generate random points and compare the two sides of the Erdős-Mordell-Barrow inequality.



We can visualize the inequality by generating random points inside the triangle and plotting the points with a color that indicates the inequality gap, darker blue corresponding to a larger gap.

www.johndcook.com/blog/2025/09/11/a-triangle-inequality-by-erdos/

And a problem of Joseph Wolstenholme

A BOOK

OF

MATHEMATICAL PROBLEMS

ON SUBJECTS INCLUDED IN

THE CAMBRIDGE COURSE.

DEVISED AND ARRANGED BY

JOSEPH WOLSTENHOLME, 1829 - 18

FELLOW OF CHRIST'S COLLEGE; SOMETIME FELLOW OF ST JOHN'S COLLEGE;
AND LATELY LECTURER IN MATHEMATICS AT CHRIST'S COLLEGE.

Joseph Wolstenholme (1829 – 1891)

Wolstenholme's *Mathematical Problems* #324

III. *Inequalities.*

322. Prove that $\cot\frac{\theta}{2} > 1 + \cot\theta$ for values of θ between 0 and π.

323. Prove that, for real values of x, $\frac{x^2 - 2x\cos\alpha + 1}{x^2 - 2x\cos\beta + 1}$ lies between $\frac{1-\cos\alpha}{1-\cos\beta}$, and $\frac{1+\cos\alpha}{1+\cos\beta}$.

324. If x, y, z be any real quantities, and A, B, C the angles of a triangle,

$$x^2 + y^2 + z^2 > 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

unless $$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}.$$

Erdős #3740: Solution by Louis Mordell I

3740 [1935, 396]. *Proposed by Paul Erdös, The University, Manchester, England.*

From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

I. *Solution by L. J. Mordell, The University, Manchester, England.*

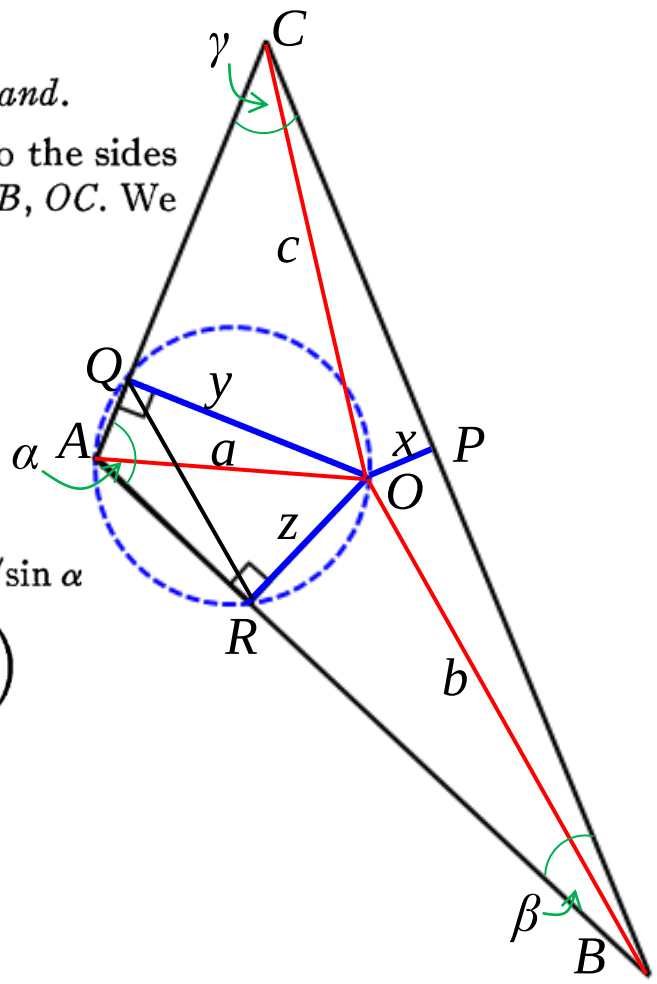
Denote by x, y, z the lengths of the perpendiculars OP, OQ, OR to the sides BC, CA, AB of the triangle ABC, and by a, b, c the lengths of OA, OB, OC. We then have

$$QR = (y^2 + z^2 + 2yz\cos\alpha)^{1/2}, \qquad a = QR/\sin\alpha, \tag{1}$$

where α, β, γ are the angles of the triangle. We have now in turn

$$\begin{aligned} a + b + c &= \sum_{\alpha,\beta,\gamma} (y^2 + z^2 + 2yz\cos\alpha)^{1/2}/\sin\alpha \\ &= \sum [(y\sin\gamma + z\sin\beta)^2 + (y\cos\gamma - z\cos\beta)^2]^{1/2}/\sin\alpha \\ &\geqq \sum (y\sin\gamma + z\sin\beta)/\sin\alpha = \sum x\left(\frac{\sin\beta}{\sin\gamma} + \frac{\sin\gamma}{\sin\beta}\right) \qquad (2) \\ &\geqq 2(x + y + z). \end{aligned}$$

This is the desired result.



The American Mathematical Monthly, Vol. 44, No. 4 (Apr., 1937), pp. 252–254

Louis Mordell II

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Denote by x, y, z the lengths of the perpendiculars OP, OQ, OR to the sides BC, CA, AB of the triangle ABC, and by a, b, c the lengths of OA, OB, OC. We then have

$$\cos(\alpha) = -\cos(\tau/2 - \alpha)$$

(1) $$QR = (y^2 + z^2 + 2yz\cos\alpha)^{1/2}, \qquad a = QR/\sin\alpha,$$

where α, β, γ are the angles of the triangle. We have now in turn

$$a + b + c = \sum_{\alpha,\beta,\gamma} (y^2 + z^2 + 2yz\cos\alpha)^{1/2}/\sin\alpha$$

$$= \sum [(y\sin\gamma + z\sin\beta)^2 + (y\cos\gamma - z\cos\beta)^2]^{1/2}/\sin\alpha \tag{2}$$

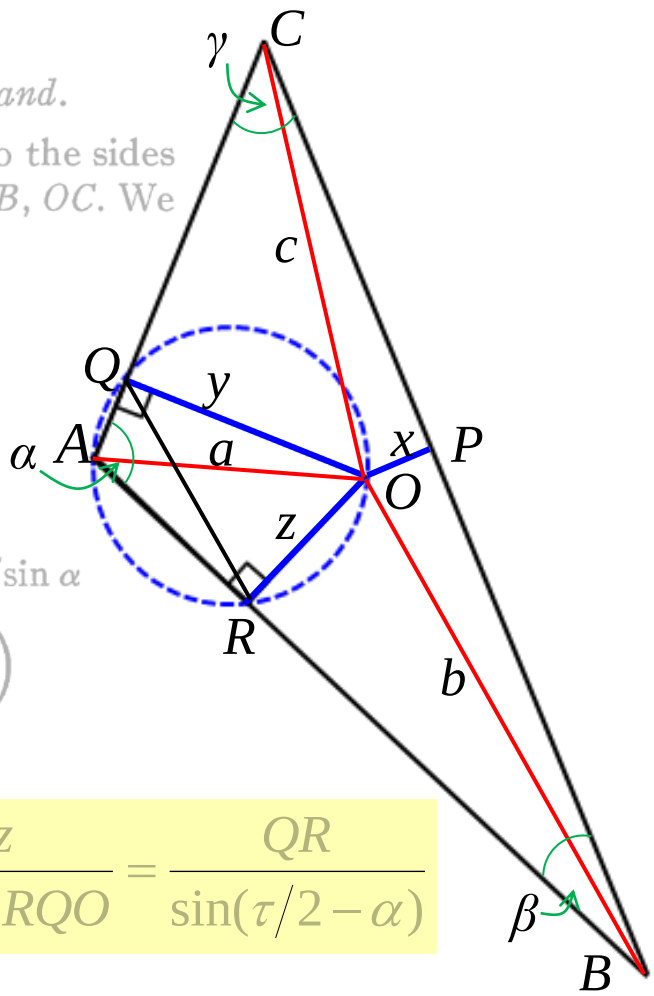
$$\geqq \sum (y\sin\gamma + z\sin\beta)/\sin\alpha = \sum x\left(\frac{\sin\beta}{\sin\gamma} + \frac{\sin\gamma}{\sin\beta}\right)$$

$$\geqq 2(x + y + z).$$

This is the desired result.

Quadrilateral $OQAR$ is cyclic.
So $\angle RQO = \angle RAO$

$$\sin\angle RQO = \sin\angle RAO = \frac{z}{a} \text{ and, sine rule : } \frac{z}{\sin\angle RQO} = \frac{QR}{\sin(\tau/2 - \alpha)}$$



Louis Mordell III

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From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

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$$a + b + c = \sum_{\alpha,\beta,\gamma} (y^2 + z^2 + 2yz\cos\alpha)^{1/2}/\sin\alpha$$

$$= \sum [(y\sin\gamma + z\sin\beta)^2 + (y\cos\gamma - z\cos\beta)^2]^{1/2}/\sin\alpha$$

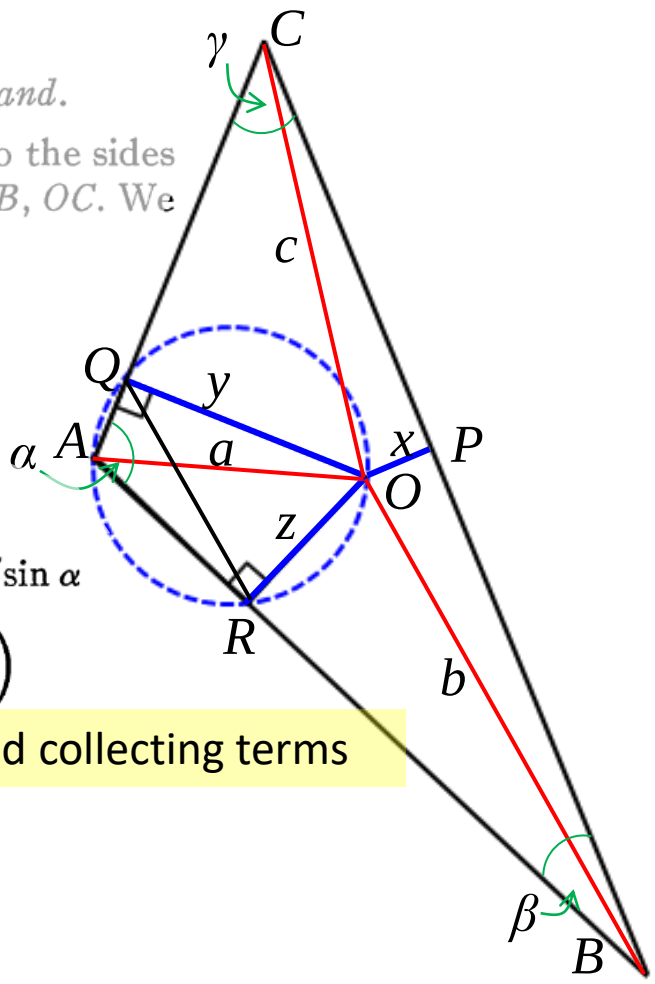
$$(2) \qquad \geqq \sum (y\sin\gamma + z\sin\beta)/\sin\alpha = \sum x\left(\frac{\sin\beta}{\sin\gamma} + \frac{\sin\gamma}{\sin\beta}\right)$$

$$\geqq 2(x + y + z).$$

This is the desired result.

Verify by expanding out

Verify by writing summation in full and collecting terms



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Louis Mordell IV

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This is the desired result.

AM - GM inequality :

$$\frac{X+Y}{2} \geq \sqrt{XY}, \; X, Y \geq 0$$

$$(X+Y)^2 \geq 4XY$$

$$X^2 + Y^2 \geq 2XY$$

$$\frac{X^2 + Y^2}{XY} \geq 2$$

$$\frac{\sin^2\beta + \sin^2\gamma}{\sin\beta\sin\gamma} \geq 2$$

Detour: AM-GM inequality

AM - GM inequality :

$$\frac{X+Y}{2} \geq \sqrt{XY},\ X, Y \geq 0$$

$$(X+Y)^2 \geq 4XY$$

$$X^2 + Y^2 \geq 2XY$$

$$\frac{X^2+Y^2}{XY} \geq 2$$

The AM-GM Inequality is Equivalent to the Bernoulli Inequality

LECH MALIGRANDA

Math. Intelligencer, Vol. 34, 2012, pp. 1–2

The AM-GM (= arithmetic-mean/geometric-mean) inequality is sometimes called the *Cauchy inequality* (1821):

$$A_n = \frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdot \ldots \cdot x_n)^{1/n} = G_n \quad (\text{all } x_k > 0). \tag{AG}$$

In the books [1] and [2] there are given 52 and 74 proofs of this inequality, respectively, and mathematicians still find new proofs (!).

Much earlier, Isaac Barrow (1670) and Jacob Bernoulli (1689) proved the inequality, which now has the name *Bernoulli inequality*:

$$x^n \geq 1 + n(x-1) \quad \text{for any } x > 0 \text{ and } n \in \mathbb{N}. \tag{B}$$

In the paper [5] we can read about the history, generalizations, and different proofs of the Bernoulli inequality.

In this note I show that these inequalities are in fact equivalent, that is, each follows one from the other. This was observed in [4, Thm 3] (see also [2], p. 213, and [5, Thm 10]), but I present here even simpler arguments.

THEOREM *Inequalities* (AG) *and* (B) *are equivalent.*

Louis Mordell V

I. *Solution by L. J. Mordell, The University, Manchester, England.*

Denote by x, y, z the lengths of the perpendiculars OP, OQ, OR to the sides BC, CA, AB of the triangle ABC, and by a, b, c the lengths of OA, OB, OC. We then have

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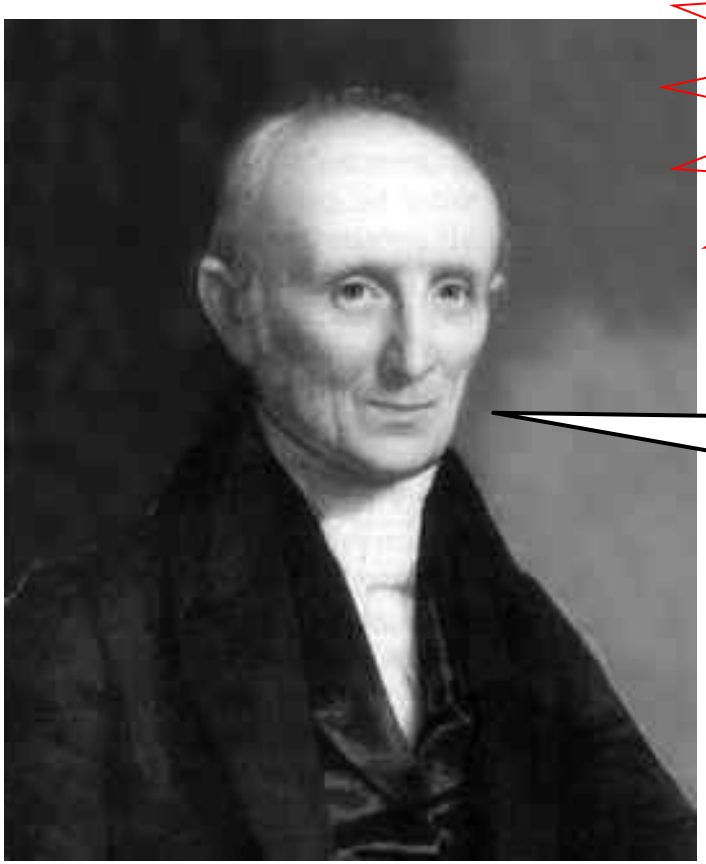
This is the desired result.



I never came across one of Laplace's "Thus it plainly appears" without feeling sure that I have many hours of hard work before me.

Nathaniel Bowditch (1773 – 1838)

Solution by David F. Barrow I

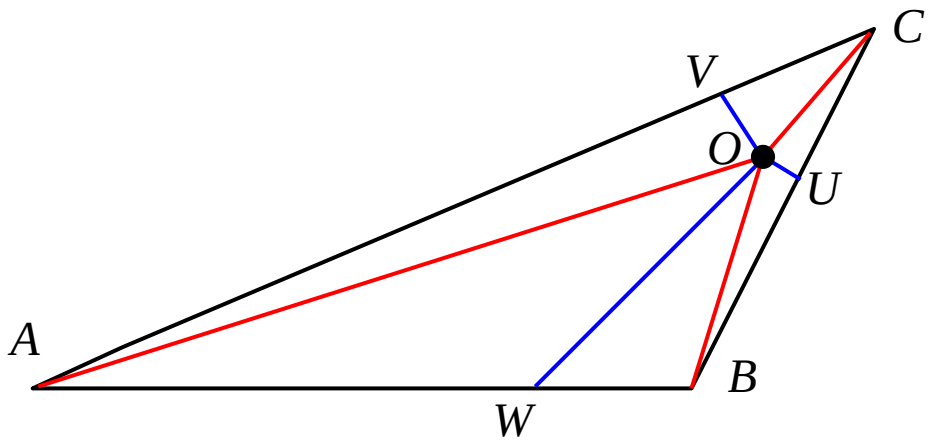
II. *Solution by David F. Barrow, University of Georgia.*

The perpendiculars are shorter than any other lines drawn from O sides of the triangle, so the proposed inequality will be proved if we establish the more stringent one of the following

THEOREM. *Twice the sum of the bisectors of the three angles formed by joining any point inside a triangle to the vertices is less than or equal to the sum of the distances of the point from the vertices.*

Let ABC be the given triangle, O the interior point, and U, V, W the points where the bisectors of angles BOC, COA, AOB cut the sides BC, CA, AB respectively. Then we wish to prove

(1) $$2(OU + OV + OW) \leqq OA + OB + OC.$$



David F. Barrow II

LEMMA 1. *If α, β, γ are real angles subject to the condition*

$$\alpha + \beta + \gamma = 180^\circ, \tag{2}$$

and a, b, c are positive constants, then

$$a \cos \alpha + b \cos \beta + c \cos \gamma \leqq \frac{ab}{2c} + \frac{bc}{2a} + \frac{ca}{2b}. \tag{3}$$

To prove this, we consider the left member of (3) as a function of two real, independent variables, α, β, and examine this function for maxima. The work is a little tedious but follows standard lines, and the result is that for certain values of a, b, c the maximum of our function is the right member of (3), while for other values of a, b, c the maximum value of the function will be one of the three expressions:

$$a + b - c, \qquad a - b + c, \qquad -a + b + c.$$

The right member of (3) exceeds each of these expressions: indeed it exceeds the first by $(-ab+bc+ca)^2/2abc$. Hence our lemma is proved.

David F. Barrow III

LEMMA 1. *If α, β, γ are real angles subject to the condition*

(2) $$\alpha + \beta + \gamma = 180°,$$

and a, b, c are positive constants, then

(3) $$a \cos \alpha + b \cos \beta + c \cos \gamma \leqq \frac{ab}{2c} + \frac{bc}{2a} + \frac{ca}{2b}.$$

To prove this, we consider the left member of (3) as a function of two real, independent variables, α, β, and examine this function for maxima. The work is a little tedious but follows standard lines, and the result is that for certain values of a, b, c the maximum of our function is the right member of (3), while for other values of a, b, c the maximum value of the function will be one of the three expressions:

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$$a\cos\alpha + b\cos\beta + c\cos\gamma \leq \frac{ab}{2c} + \frac{bc}{2a} + \frac{ca}{2b}$$

$$= \frac{(ab)^2 + (bc)^2 + (ca)^2}{2abc}$$

$$ab \cdot ac\cos\alpha + ab \cdot bc\cos\beta + ac \cdot bc\cos\gamma \leq \frac{1}{2}\left((ab)^2 + (bc)^2 + (ca)^2\right)$$

$$zx\cos\alpha + xy\cos\beta + yz\cos\gamma = \frac{1}{2}\left(x^2 + y^2 + y^2\right)$$

$$\text{with } x = ab, y = bc, z = ca$$

Wolstenholme meets Erdős I

III. *Inequalities.*

322. Prove that $\cot\frac{\theta}{2} > 1 + \cot\theta$ for values of θ between 0 and π.

323. Prove that, for real values of x, $\frac{x^2 - 2x\cos\alpha + 1}{x^2 - 2x\cos\beta + 1}$ lies between $\frac{1-\cos\alpha}{1-\cos\beta}$, and $\frac{1+\cos\alpha}{1+\cos\beta}$.

324. If x, y, z be any real quantities, and A, B, C the angles of a triangle,

$$x^2 + y^2 + z^2 > 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

unless

$$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}.$$

$$ab \cdot ac\cos\alpha + ab \cdot bc\cos\beta + ac \cdot bc\cos\gamma \le \frac{1}{2}\left((ab)^2 + (bc)^2 + (ca)^2\right)$$

$$zx\cos\alpha + xy\cos\beta + yz\cos\gamma \le \frac{1}{2}\left(x^2 + y^2 + y^2\right)$$

with $x = ab, y = bc, z = ca$

David F. Barrow IV

LEMMA 2. *The following algebraic identity may be verified directly*

$$\frac{2x^2(y+z)}{(x+y)(z+x)}+\frac{2y^2(z+x)}{(x+y)(y+z)}+\frac{2z^2(x+y)}{(y+z)(z+x)} \tag{4}$$
$$\equiv x+y+z-\frac{xy(x-y)^2+yz(y-z)^2+zx(z-x)^2}{(x+y)(y+z)(z+x)}.$$

LEMMA 3. *In any triangle the bisector of an angle equals the cosine of half this angle multiplied by twice the product of the including sides and divided by the sum of the including sides.*

For instance in triangle BOC, the bisector OU is given by

$$OU=\frac{2(OB)(OC)}{OB+OC}\cos\tfrac{1}{2}(BOC). \tag{5}$$

In deriving this the writer made a roundabout use of trigonometry and the fact that the bisector divides the opposite side in segments proportional to the adjacent sides. It looks as if there should be some shorter way to get it.

Note by the Editor. Formula (5) is given in Loney's *Plane Trigonometry*, p. 247.

David F. Barrow V

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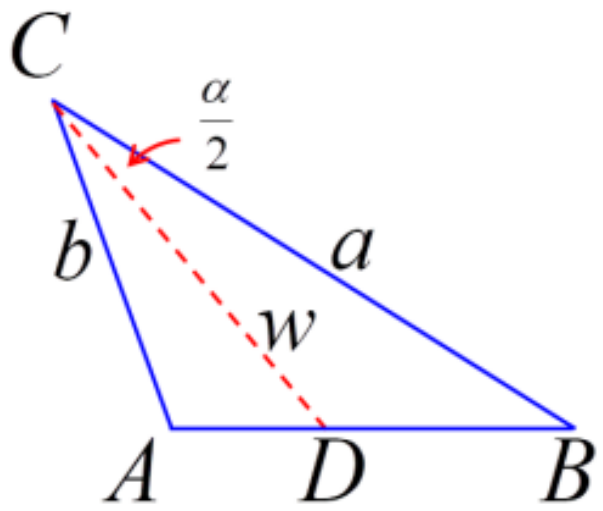
Angle Bisection Theorem
In the triangle on points A, B, C with opposite sides a, b, c, let the bisector of the angle at C meet the opposite side at D. Then

$$\frac{|AD|}{|BD|} = \frac{b}{a}.$$



Corollary *In the theorem, let the angle at C be α. Then the length of the bisecting line CD is given by:*

$$w = \frac{2ab}{a+b} \cos\left(\frac{\alpha}{2}\right).$$

S. L. Loney

4 languages

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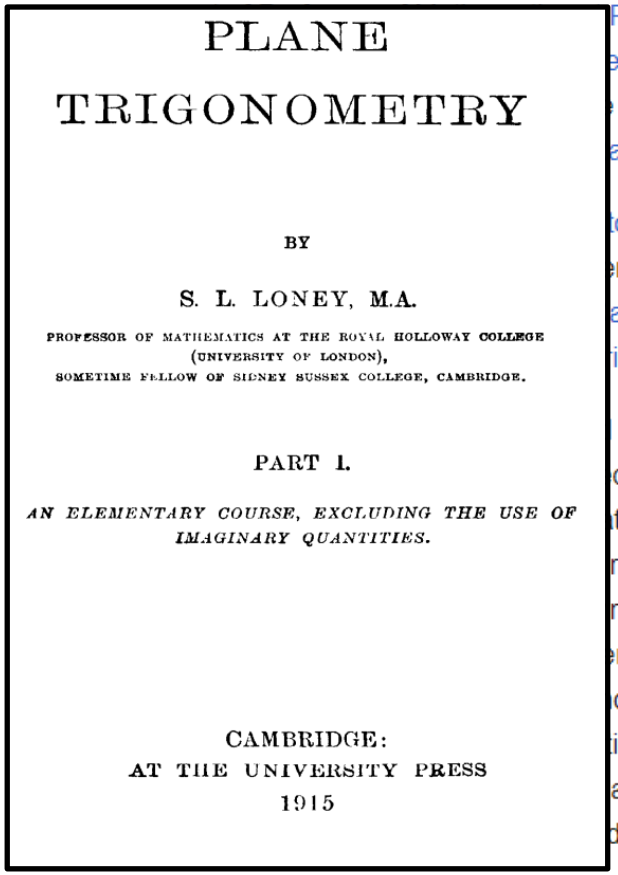
From Wikipedia, the free encyclopedia


PLANE
TRIGONOMETRY

BY

S. L. LONEY, M.A.

PROFESSOR OF MATHEMATICS AT THE ROYAL HOLLOWAY COLLEGE
(UNIVERSITY OF LONDON),
SOMETIME FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

PART I.

AN ELEMENTARY COURSE, EXCLUDING THE USE OF IMAGINARY QUANTITIES.

CAMBRIDGE:
AT THE UNIVERSITY PRESS
1915

Sydney Luxton Loney, M.A. (16 March 1860 – 16 May 1939) held the esteemed Royal Holloway College, Egham, Surrey, and was ege, Cambridge. He authored several mathematics into multiple reprints over the years. He is known amanujan.[1]

tone Grammar School, then moved to Tonbridge matics first became evident. In 1882 he graduated ambridge as 3rd Wrangler, placing him third in the ipos.

a Fellow of Sidney Sussex College from 1885 to d his engagement with both teaching and research. thematics at Royal Holloway College (University of retirement in 1920. Beyond his professorship, nance: he became a Senator of the University of rnor of Royal Holloway in 1920, Chairman of the d Deputy Chairman of its Court in 1929. Locally, he ion Committee from 1909 to 1937, was Mayor of as a Justice of the Peace, demonstrating a d academia.

S. L. Loney

Born	March 16, 1860 Chevithorne, Devon, England
Died	May 16, 1939 (aged 79) Richmond, London
Alma mater	Sidney Sussex College, Cambridge
Occupations	Professor, author
Organization	Royal Holloway College

Loney's *Plane Trigonometry* and *The Elements of Coordinate Geometry* have remained staples in Indian senior-high curricula and engineering-entrance coaching, prized for their lucid theory and graduated exercises that build problem-solving skills. Perhaps most notably, an eleven-year-old Srinivasa Ramanujan borrowed *Plane Trigonometry* in 1899 and, working through it rigorously over two years, encountered his first substantial piece of formal mathematics outside his school syllabus, a pivotal step in his self-education.[2]

David F. Barrow VI

LEMMA 3. *In any triangle the bisector of an angle equals the cosine of half this angle multiplied by twice the product of the including sides and divided by the sum of the including sides.*

For instance in triangle *BOC*,

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that the bisector divides the op
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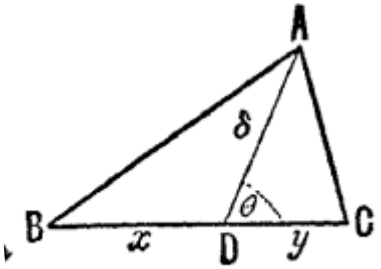
218. Bisectors of the angles.



Also, if δ be the length of AD and θ the angle it makes with BC, we have

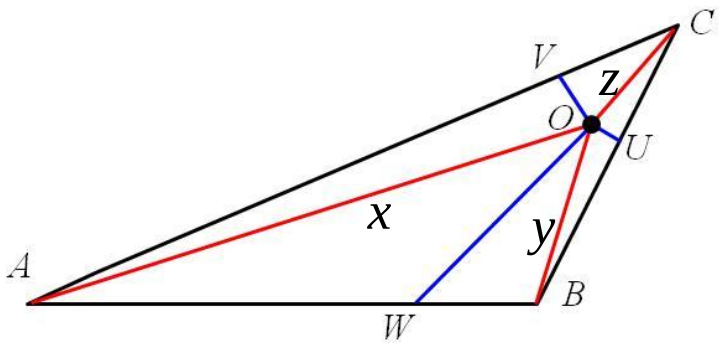
$$\triangle ABD + \triangle ACD = \triangle ABC.$$

$$\therefore \quad \frac{1}{2} c\delta \sin\frac{A}{2} + \frac{1}{2} b\delta \sin\frac{A}{2} = \frac{1}{2} bc \sin A,$$

i.e.

$$\delta = \frac{bc}{b+c} \frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{b+c} \cos\frac{A}{2} \quad \ldots\ldots\ldots (2).$$

David F. Barrow VII



$$2(OU + OV + OW) \leqq OA + OB + OC.$$

LEMMA 1. *If α, β, γ are real angles subject to the condition*

$$\alpha + \beta + \gamma = 180^\circ, \tag{2}$$

and a, b, c are positive constants, then

$$a\cos\alpha + b\cos\beta + c\cos\gamma \leqq \frac{ab}{2c} + \frac{bc}{2a} + \frac{ca}{2b}. \tag{3}$$

LEMMA 3. *In any triangle the bisector of an angle equals th angle multiplied by twice the product of the including sides and of the including sides.*

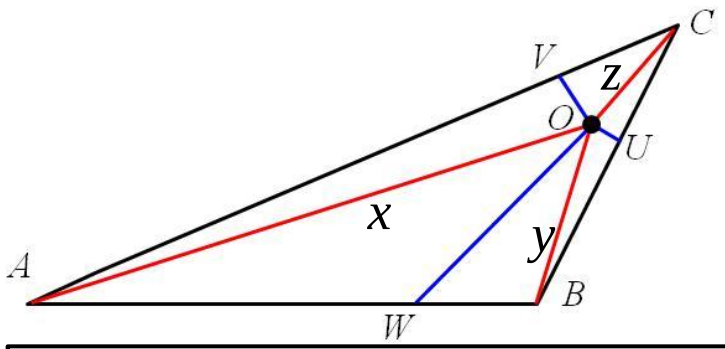
For instance in triangle BOC, the bisector OU is given by

$$OU = \frac{2(OB)(OC)}{OB + OC}\cos\tfrac{1}{2}(BOC). \tag{5}$$

Proof of the theorem: Let angles BOC, COA, AOB be denoted by 2α, 2β, 2γ respectively. Then since these angles form a perigon, it is evident that α, β, γ satisfy (2). Also let OA, OB, OC be denoted by x, y, z respectively. Then by use of lemma 3, we have

$$2(OU + OV + OW) = \frac{4yz}{y+z}\cos\alpha + \frac{4zx}{z+x}\cos\beta + \frac{4xy}{x+y}\cos\gamma. \tag{6}$$

David F. Barrow VII



LEMMA 1. *If α, β, γ are real angles subject to the condition*

(2) $$\alpha + \beta + \gamma = 180°,$$

and a, b, c are positive constants, then

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Then by making use of (3) of lemma (1), with obvious substitutions for a, b, c, we find that the right member of (6) is not greater than the left member of (4), and this latter identity makes the theorem immediately obvious.

COROLLARY. *The equality sign in* (1) *holds only when ABC is an equilateral triangle and O is its center.*

LEMMA 2. *The following algebraic identity may be verified directly*

(4) $$\frac{2x^2(y+z)}{(x+y)(z+x)} + \frac{2y^2(z+x)}{(x+y)(y+z)} + \frac{2z^2(x+y)}{(y+z)(z+x)} \equiv x + y + z - \frac{xy(x-y)^2 + yz(y-z)^2 + zx(z-x)^2}{(x+y)(y+z)(z+x)}.$$

Erdős meets Wolstenholme II

324. If x, y, z be any real quantities, and A, B, C the angles of a triangle,

$$x^2 + y^2 + z^2 > 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

unless $\dfrac{x}{\sin A} = \dfrac{y}{\sin B} = \dfrac{z}{\sin C}$.

3740. *Proposed by Paul Erdös*

From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

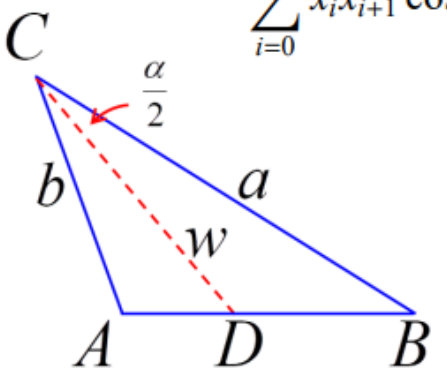
$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

Wolstenholme's inequality *Let $x_0, \ldots, x_n$ and $\theta_0, \ldots, \theta_{n-1}$ be positive real numbers satisfying $x_n = x_0$ and $\theta_0 + \ldots + \theta_{n-1} = \tau$. Then*

$$\sum_{i=0}^{n-1} x_i x_{i+1} \cos(\theta_i/2) \leq \cos(\tau/2n) \sum_{i=0}^{n-1} x_i^2.$$

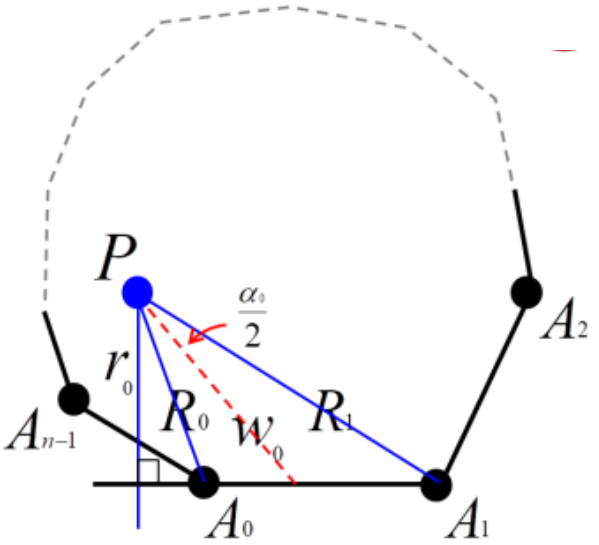
Angle Bisection Theorem
In the triangle on points A, B, C with opposite sides a, b, c, let the bisector of the angle at C meet the opposite side at D. Then

$$\frac{|AD|}{|BD|} = \frac{b}{a}.$$



Corollary *In the theorem, let the angle at C be α. Then the length of the bisecting line CD is given by:*

$$w = \frac{2ab}{a+b}\cos\left(\frac{\alpha}{2}\right).$$



Our illustration shows the application of the inequality (and some textbook geometry) to prove **The Erdős–Mordell–Barrow inequality**: *Let P be a point interior to a convex n-vertex polygon. If $R_i, i = 0, \ldots, n-1$, are the distances from P to the polygon vertices, and $r_i, i = 0, \ldots, n-1$, are the perpendicular distances from P to the (extended) polygon edges, then $\sum r_i \leq \cos(\tau/n) \sum R_i$.*

Indeed, the Corollary follows from the Angle Bisection Theorem by calculating the values of $|AD|$ and $|BD|$ using the Cosine Rule. Now in Erdős–Mordell–Barrow, let w_i be the distance from P to the edge A_iA_{i+1} along the line that bisects angle $\alpha_i = \angle A_iPA_{i+1}$. Observe that $r_i \leq w_i$. And now in our Corollary we will apply the AM-GM inequality: $(R_i + R_{i+1})/2 \leq \sqrt{R_iR_{i+1}}$. We get $r_i \leq w_i = 2\dfrac{R_iR_{i+1}}{R_i + R_{i+1}} \cos\alpha_i/2 \leq \sqrt{R_iR_{i+1}} \cos\alpha_i/2$. Since $\sum \alpha_i = \tau$ we may apply Wolstenholme:

$$\sum r_i \leq \sum w_i \leq \sum \sqrt{R_i}\sqrt{R_{i+1}} \cos(\alpha_i/2) \leq \cos(\tau/2n) \sum \left(\sqrt{R_i}\right)^2 = \cos(\tau/2n) \sum R_i.$$

In a regular n-gon the distance R from the centre to a vertex is related to the perpendicular distance r from the centre to an edge by $R = r/\cos(\tau/2n)$. So the regular polygons and their centres give equality in Erdős–Mordell–Barrow and provide an example which achieves equality in Wolstenholme.

Thanks for listening, more at
www.theoremoftheday.org/Theorems.html#270

THEOREM OF THE DAY

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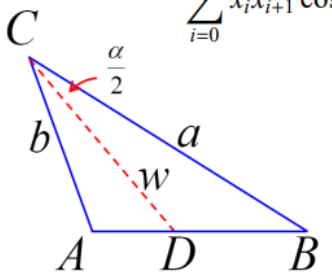

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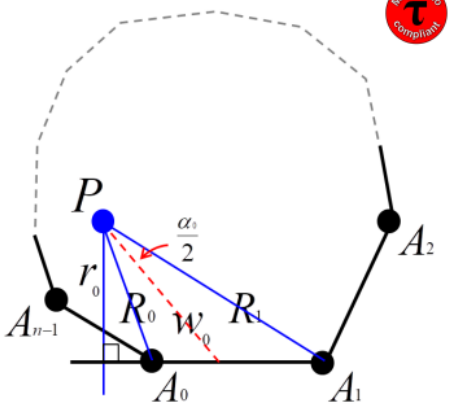

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Joseph Wolstenholme offered the case $n = 3$ of his inequality in a mathematical problem collection published in 1867. It has become a canonical example of a whole range of triangle inequalities and geometrical inequalities more generally. The extension to arbitrary n appeared a century later presumably in response to the challenge of generalising Erdős–Mordell–Barrow; which in turn started life as a triangle problem posed by Paul Erdős in 1935, solved two years later by the number theorist L. J. Mordell and independently by D.F. Barrow (whose solution introduced the idea of using angle bisectors).

Web links: cut-the-knot.org/triangle/ErdosMordell.shtml
Further reading: *Charming Proofs: A Journey into Elegant Mathematics* by Claudi Alsina and Roger B. Nelsen, Mathematical Association of America, 2011, Chapter 5.