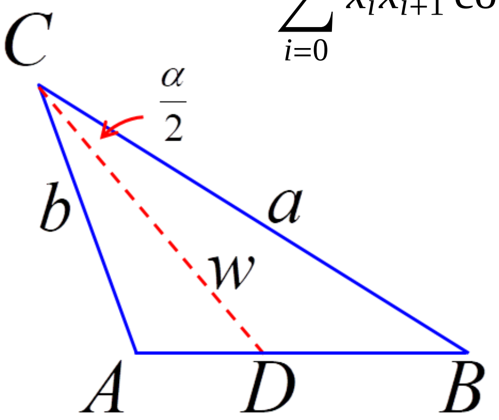
THEOREM OF THE DAY

Wolstenholme's inequality *Let $x_0, \ldots, x_n$ and $\theta_0, \ldots, \theta_{n-1}$ be positive real numbers satisfying $x_n = x_0$ and $\theta_0 + \ldots + \theta_{n-1} = \tau$. Then*

$$\sum_{i=0}^{n-1} x_i x_{i+1} \cos(\theta_i/2) \leq \cos\tau/(2n) \sum_{i=0}^{n-1} x_i^2.$$

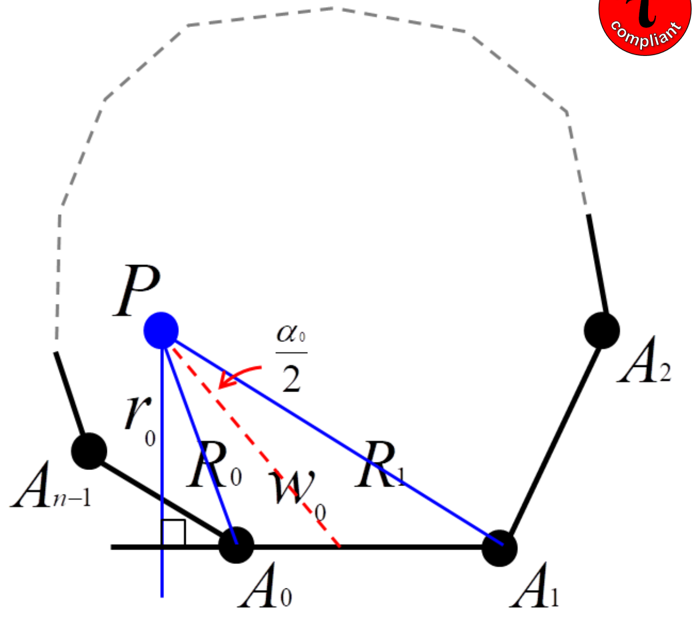
Angle Bisection Theorem *In the triangle on points A, B, C with opposite sides a, b, c, let the bisector of the angle at C meet the opposite side at D. Then*

$$\frac{|AD|}{|BD|} = \frac{b}{a}.$$



Corollary *In the theorem, let the angle at C be α. Then the length of the bisecting line CD is given by:*

$$w = \frac{2ab}{a+b}\cos\left(\frac{\alpha}{2}\right).$$



Our illustration shows the application of the inequality (and some textbook geometry) to prove **The Erdős–Mordell–Barrow inequality**: *Let P be a point interior to a convex n-vertex polygon. If R_i, $i = 0, \ldots, n-1$, are the distances from P to the polygon vertices, and r_i, $i = 0, \ldots, n-1$, are the perpendicular distances from P to the (extended) polygon edges, then $\sum r_i \leq \cos(\tau/n) \sum R_i$.*

Indeed, the Corollary follows from the Angle Bisection Theorem by calculating the values of $|AD|$ and $|BD|$ using the Cosine Rule. Now in Erdős–Mordell–Barrow, let w_i be the distance from P to the edge A_iA_{i+1} along the line that bisects angle $\alpha_i = \angle A_iPA_{i+1}$. Observe that $r_i \leq w_i$. And now in our Corollary we will apply the AM-GM inequality: $(R_i + R_{i+1})/2 \leq \sqrt{R_iR_{i+1}}$. We get $r_i \leq w_i = 2\dfrac{R_iR_{i+1}}{R_i + R_{i+1}} \cos\alpha_i/2 \leq \sqrt{R_iR_{i+1}} \cos\alpha_i/2$. Since $\sum \alpha_i = \tau$ we may apply Wolstenholme:

$$\sum r_i \leq \sum w_i \leq \sum \sqrt{R_i}\sqrt{R_{i+1}}\cos(\alpha_i/2) \leq \cos(\tau/2n) \sum \left(\sqrt{R_i}\right)^2 = \cos(\tau/2n) \sum R_i.$$

In a regular n-gon the distance R from the centre to a vertex is related to the perpendicular distance r from the centre to an edge by $R = r/\cos(\tau/2n)$. So the regular polygons and their centres give equality in Erdős–Mordell–Barrow and provide an example which achieves equality in Wolstenholme.

Joseph Wolstenholme offered the case $n = 3$ of his inequality in a mathematical problem collection published in 1867. It has become a canonical example of a whole range of triangle inequalities and geometrical inequalities more generally. The extension to arbitrary n appeared a century later presumably in response to the challenge of generalising Erdős–Mordell–Barrow; which in turn started life as a triangle problem posed by Paul Erdős in 1935, solved two years later by the number theorist L. J. Mordell and independently by D.F. Barrow (whose solution introduced the idea of using angle bisectors).

Web links: cut-the-knot.org/triangle/ErdosMordell.shtml

Further reading: *Charming Proofs: A Journey into Elegant Mathematics* by Claudi Alsina and Roger B. Nelsen, Mathematical Association of America, 2011, Chapter 5.