THEOREM OF THE DAY

The First Isomorphism Theorem Let $G$ and $H$ be groups and $f : G \to H$ a homomorphism of $G$ to $H$ with image $\text{Im}(f)$ and kernel $\text{ker}(f)$. Then $G/\text{ker}(f)$ and $\text{Im}(f)$ are isomorphic groups:

$$G/\text{ker}(f) \cong \text{Im}(f).$$

**EXAMPLE**

Complex numbers can be thought of as points in the plane (the Argand diagram). The set $\mathbb{C}^*$ of all points apart from the origin $(0,0)$ can be

(A) represented uniquely as a pair (distance $r$ from origin, angle $\theta$ from real axis);

(B) multiplied together by the rule

$$(r, \theta) \times (s, \phi) = (r \times s, \theta + \phi);$$

(C) mapped onto the positive real numbers by the function

$$P : (r, \theta) \mapsto r^2$$

(we write $\text{Im}(P) = \mathbb{R}^{>0}$).

The function $P$ in (C) is illustrated in the above plot. It is a homomorphism because it preserves the multiplication defined in (B):

$$P((r, \theta) \times (s, \phi)) = (r \times s)^2 = r^2 \times s^2 = P((r, \theta)) \times P((s, \phi)).$$

The unit circle constitutes precisely the set of points mapped to 1 by $P$. For any mapping $f$ between groups, the set of elements mapped to the identity is called the *kernel* of $f$, written $\text{ker}(f)$. So $\text{ker}(P)$ is the unit circle.

A group consists of a set together with a multiplication rule for which division ‘works’ in the expected way. Distinguishing and classifying groups is of great importance in group theory; the so-called isomorphism theorems were first identified by Emmy Noether as a basic tool for this task. Here, two apparently different groups are revealed as identical by the First Isomorphism Theorem.

Web links: [www.math.uic.edu/~radford/math516f06/IsoThms.pdf](http://www.math.uic.edu/~radford/math516f06/IsoThms.pdf); the ripples image is by Jon Woodring.