THEOREM OF THE DAY

The First Isomorphism Theorem *Let G and H be groups and $f : G \to H$ a homomorphism of G to H with image* $\mathrm{Im}(f)$ *and kernel* $\ker(f)$. *Then* $G/\ker(f)$ *and* $\mathrm{Im}(f)$ *are isomorphic groups:*

$$G/\ker(f) \cong \mathrm{Im}(f).$$

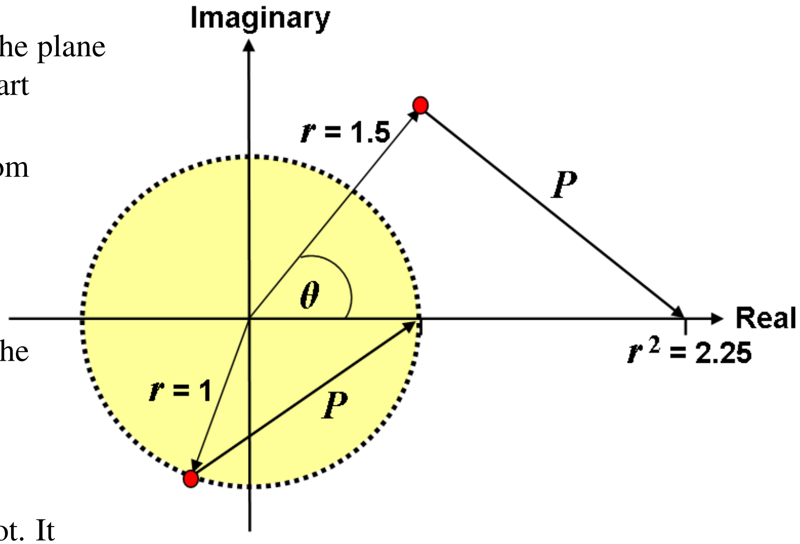
EXAMPLE

Complex numbers can be thought of as points in the plane (the Argand diagram). The set $\mathbb{C}^*$ of all points apart from the origin $(0, 0)$ can be

(A) represented uniquely as a pair (distance r from origin, angle θ from real axis);

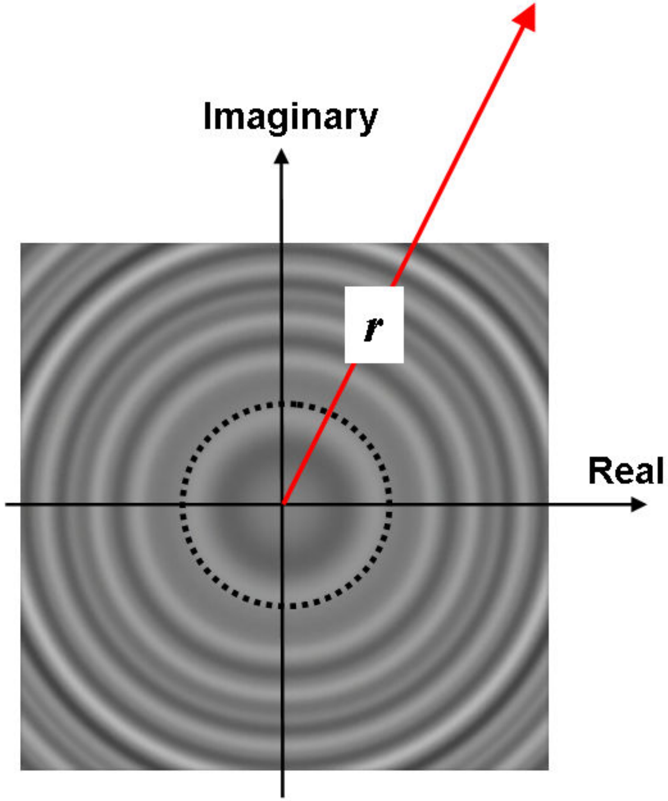
(B) multiplied together by the rule
$$(r, \theta) \times (s, \phi) = (r \times s, \theta + \phi);$$

(C) mapped onto the positive real numbers by the function
$$P : (r, \theta) \mapsto r^2$$
(we write $\mathrm{Im}(P) = \mathbb{R}^{>0}$).



The function P in (C) is illustrated in the above plot. It is a homomorphism because it preserves the multiplication defined in (B):
$$P((r, \theta) \times (s, \phi)) = (r \times s)^2 = r^2 \times s^2 = P((r, \theta)) \times P((s, \phi)).$$

The unit circle constitutes precisely the set of points mapped to 1 by P. For any mapping f between groups, the set of elements mapped to the identity is called the *kernel* of f, written $\ker(f)$. So $\ker(P)$ is the unit circle.



Now multiply the whole set of points in $\ker(P)$ by some point (r, θ) in the plane, using the rule defined earlier in (B). You get a new circle at radius r — the multiples of $\ker(P)$ expand out like ripples in a pond as r increases (as depicted above). These ripples are the *cosets* of $\ker(P)$; the whole collection is denoted $\mathbb{C}^*/\ker(P)$ and its members multiply exactly in the same manner as the positive reals. Thus, $\mathbb{C}^*/\ker(P) \cong \mathbb{R}^{>0}$, as indeed the First Isomorphism Theorem guarantees.

A group consists of a set together with a multiplication rule for which division 'works' in the expected way. Distinguishing and classifying groups is of great importance in group theory; the so-called isomorphism theorems were first identified by Emmy Noether as a basic tool for this task. Here, two apparently different groups are revealed as identical by the First Isomorphism Theorem.

Web links: www.math.uic.edu/~radford/math516f06/IsoThms.pdf; the ripples image is by Jon Woodring.

Further reading: *The Architecture of Modern Mathematics: Essays in History and Philosophy* by José Ferreirós and Jeremy Gray (eds.), Oxford University Press, 2006 (the chapter by Colin McLarty)