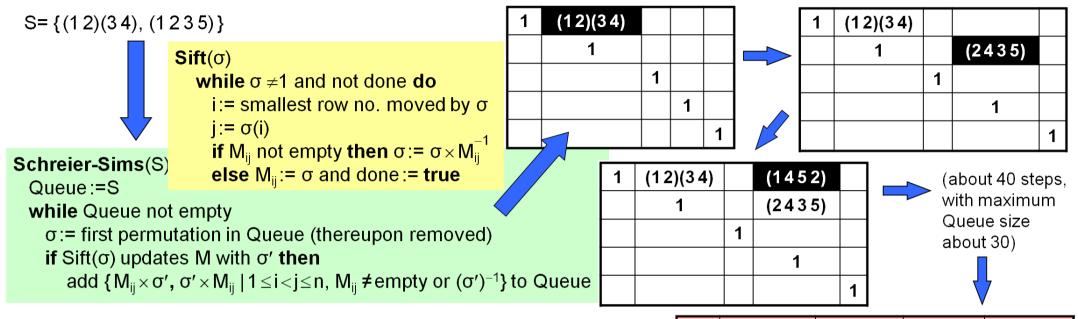
THEOREM OF THE DAY

The McIver–Neumann Half-*n* **Bound** Let Ω be a set of order $n, n \neq 3$, and let G be a permutation group acting on Ω . Then G may be generated by $\lfloor n/2 \rfloor$ elements.



The generation of a permutation group from a set of permutations is illuminated by the wonderful algorithm which Charles Sims derived from a 1927 lemma of Otto Schreier. In the illustration above, the algorithm is applied to a set of two permutations. The main work is done by the Sift routine, which first places (12)(34) into row 1 column 2 cell in the 5 × 5 table *M*. The next permutation $\sigma = (1235)$ is a candidate for the same cell since $\sigma(1) = 2$; since the cell is occupied by (12)(34) we calculate $\sigma := \sigma \times M_{12}^{-1} = (1235) \times (12)(34) = (2435)$. The new cell M_{24} is now indicated for σ because now $\sigma(2) = 4$. Subsequently, the queue is augmented in the main algorithm by pre- and post-multiplying (2435) with each non-identity entry of *M*. The queue now

1	(12)(34)	(1345)	(1452)	(15)(24)
	1	(23)(45)	(2435)	(2534)
		1	_	_
			1	_
				1

begins with $(12)(34) \times (2435) = (1452)$ which Sift places directly into position $M_{1,4}$. Although the queue tends to grow rapidly it must eventually become empty; in the current example the algorithm terminates with table shown above, bottom-right. This table describes the generated group in the following way: the order of the group is the product of the numbers of permutations in each row, in this case $5 \times 4 \times 1 \times 1 = 20$ (the group happens to be the Frobenius group of order 20, aka the Galois group of the polynomial $x^5 - 2$ over \mathbb{Q}); membership of the group is tested by running Sift on a candidate permutation σ which will be allocated to an empty table location if and only if it does not belong to the group. **Exercise:** try the algorithm with $\Omega = \{1, \ldots, 8\}$ and $S = \{(12), (34), (56), (78)\}$.

It is easy to generate large groups from a very few permutations; n/2 permutations may sometimes be necessary as the Exercise above easily shows. But Annabelle McIver and Peter M. Neumann's 1987 theorem seems to be deep and mysterious: it rests on the Classification of the Finite Simple Groups and suggests no obvious way of finding a generating set meeting its bound.

Web link: www.math.uni-bielefeld.de/~baumeist/wop2017/: click on Vortragsfolien and choose Gareth Tracey.

 Further reading: Enumeration of Finite Groups by Simon R. Blackburn, Peter M. Neumann and Geetha Venkataraman, Cambridge University Press, 2007, chapter 7.

 Created by Robin Whitty for www.theoremoftheday.org