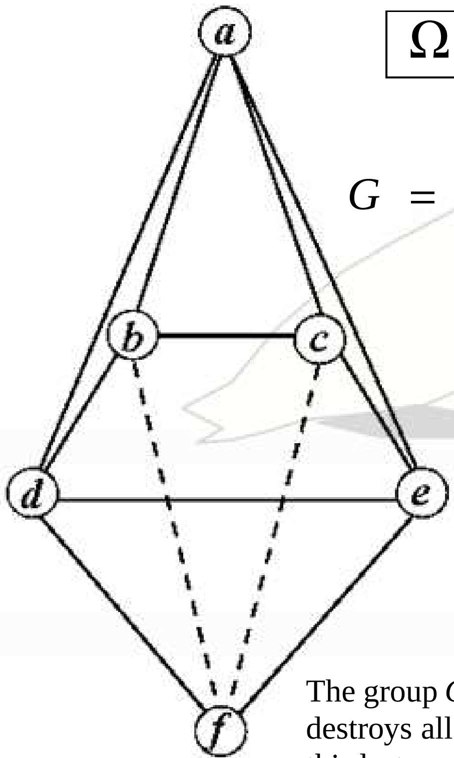
THEOREM OF THE DAY

The Cameron–Fon-Der-Flaass IBIS Theorem *Let G be a permutation group acting on a set Ω. Then the following are equivalent:*

1. *all irredundant bases of G have the same size;*
2. *the irredundant bases of G are preserved by re-ordering;*
3. *the irredundant bases of G form the bases of a matroid.*

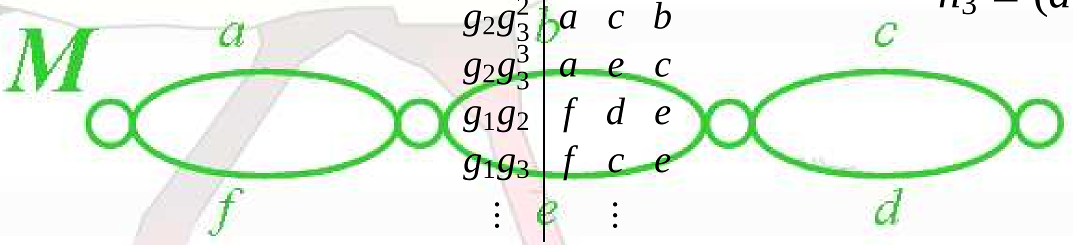


$$\Omega = \{a, b, c, d, e, f\}$$

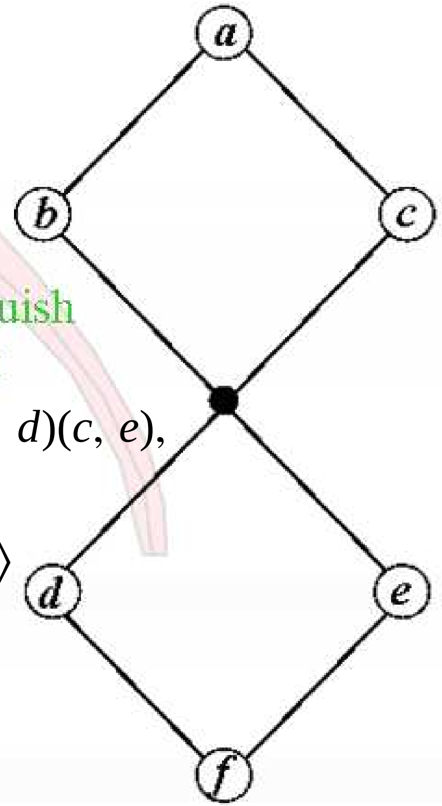
$$G = \langle g_1 = (a,\ f),\ g_2 = (b,\ d)(c,\ e),\ g_3 = (b,\ c,\ e,\ d)\,\rangle$$

	a	b	c
G 1	a	b	c
g_1	f	b	c
g_2	a	d	e
g_3	a	c	e
g_3^2	a	e	d
g_3^3	a	d	b
g_2g_3	a	b	d
$g_2g_3^2$	a	c	b
$g_2g_3^3$	a	e	c
g_1g_2	f	d	e
g_1g_3	f	c	e
⋮			⋮

The action of a group on a base is sufficient to distinguish each group element



$$H = \langle h_1 = (a,\ f)(b,\ d)(c,\ e),\ h_2 = (b,\ c),\ h_3 = (d,\ e)\,\rangle$$



The group G represents the 16 symmetries of the irregular octahedron shown on the left. A **base** is a sequence of vertices which, when fixed, destroys all symmetry. Fixing vertex a removes g_1, the vertical rotation; fixing b removes g_2 and g_3 but not $g_2g_3 = (c, d)$; fixing c removes this last symmetry: and $[a, b, c]$ is a base. Or we could finish by fixing d: so another base is $[a, b, d]$. In fact, in the graph M above-centre, any three edges which meet all three vertices, taken in any order, correspond to a base. Each of these bases is **irredundant**: each element of each sequence plays its part in removing additional symmetry. Moreover, M gives us a matroid: take any two bases B_1 and B_2 and they obey the 'exchange property' that we can replace some element in $B_1 \backslash B_2$ with an element in $B_2 \backslash B_1$ and again get a base. Not all groups have such well-behaved bases! The 8 symmetries of the figure on the right form the group H. Now $[a, b, d]$ is an irredundant base but the reordering $[b, a, d]$ is not irredundant: with b fixed, no remaining symmetries move a, so $[b, d]$ is a smaller base.

Peter Cameron and Dima Fon-Der-Flaass made this too-good-to-be-true link between group theory and combinatorics in 1995. A group satisfying its three equivalent properties is called an IBIS group: it has 'Irredundant Bases of Invariant Size'.

Web link: www-groups.mcs.st-andrews.ac.uk/~pjc/talks/whittle/pjc_wellington1.pdf

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999, chapter 4.