THEOREM OF THE DAY

Moufang’s Theorem  In a Moufang loop any three elements which associate generate a group.

The Quaternions

\[
\begin{array}{cccc}
1 & i & j & k \\
i & i & -1 & k \\
j & j & -k & -1 \\
k & k & j & -i & -1 \\
\end{array}
\]

The Octonions

\[
\begin{array}{cccccccc}
1 & 1 & i & j & k & e & f & g & h \\
i & i & -1 & k & j & e & f & g & h \\
j & j & -k & -1 & i & -g & h & e & -f \\
k & k & j & -i & -1 & h & g & f & -e \\
e & e & -f & g & -h & -1 & i & -j & k \\
f & f & e & -h & -g & -i & -1 & k & j \\
g & g & -h & -e & f & j & -k & -1 & i \\
h & h & g & f & e & -k & j & -i & -1 \\
\end{array}
\]

With \(i\) the imaginary constant whose square is \(-1\), the set \(\{\pm 1, \pm i\}\) forms a group: multiplication keeps you inside the set, it allows inverses (e.g. \(i \times -i = -i^2 = -(-1) = 1\)), and it is associative (that is, \(x \times (y \times z)\) is the same as \((x \times y) \times z\) — the bracketing can safely be forgotten). In 1843, the great Irish scientist William Rowan Hamilton discovered the quaternions: \(i\) is joined by mysterious companions \(j\) and \(k\) who multiply according to the circular diagram above left: if \(x\) and \(y\) follow each other clockwise round the circle, then \(x \times y = +\) the other quantity; if anticlockwise, the product is negative: \(ij = k, kj = -i\), etc. And, again, \(\{\pm 1, \pm i, \pm j, \pm k\}\) is a group. J.T. Graves, a professor of law in London, was inspired to try and go one better: just two months later he had produced the octonions, whose multiplication table is given centre and can be constructed from the Fano plane (above right; to keep the diagram simple, only three points from each circle are given: we must imagine \(e \rightarrow j \rightarrow g\), for example, cycling back round to \(e\)). But Hamilton spotted a snag: octonion multiplication is not associative. For example, \((ij)e = ke = h\) but \(i(je) = i(-g) = -ig = -h\). The octonions were discovered independently by Cayley and are sometimes called Cayley numbers.

A hundred years later, in Germany, Ruth Moufang invented a deep connection between algebra and projective geometry via the idea of a loop: exactly those arithmetics which fail to be groups just through being nonassociative. A Moufang loop is one in which any \(x, y\) and \(z\) nearly associate: they obey three (equivalent) Moufang identities:

left: \((xy)z = x(yz)\),  middle: \((xy)(xz) = (x(yz))x\),  right: \((xy)z = x(yz)\).

You can check these hold in the octonions which are a classic example of a Moufang loop: the quaternions, hiding associatively inside, are a group thanks to Moufang’s theorem. You can check, too, the corollary to Moufang’s Theorem, that any Moufang loop is diassociative: any pair of elements whatsoever generates a group (put \(y = 1\) in the left Moufang identity and apply the theorem to \(x, x\) and \(z\)). E.g. \((e, f)\) generate a group of order 8, with elements \(\{\pm 1, \pm e, \pm f, \pm e f\}\).

Ruth Moufang (1905–1977) played an indirect part in the classification of the finite simple groups: Richard Parker’s 1985 construction of a Moufang loop of order \(2^{13}\) was used by John Conway to construct the Monster (order \(\approx 8 \times 10^{53}\)).
