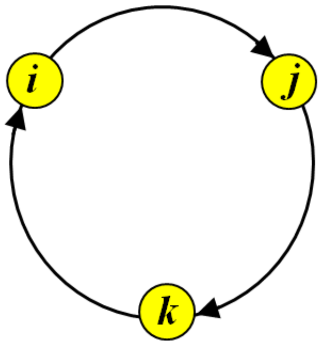
THEOREM OF THE DAY

Moufang's Theorem *In a Moufang loop any three elements which associate generate a group.*

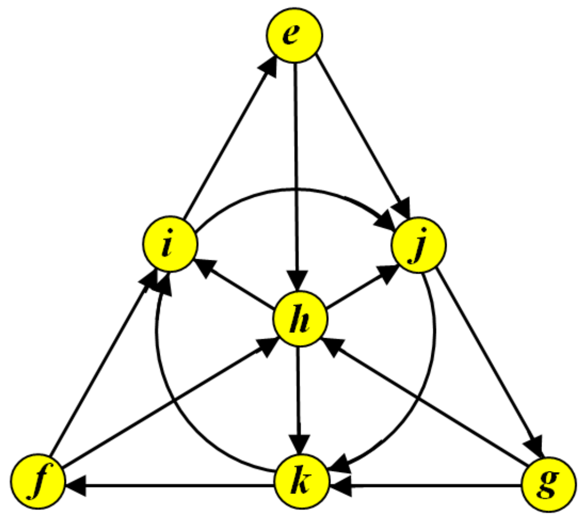
	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The Quaternions



	1	i	j	k	e	f	g	h
1	1	i	j	k	e	f	g	h
i	i	-1	k	$-j$	f	$-e$	h	$-g$
j	j	$-k$	-1	i	$-g$	h	e	$-f$
k	k	j	$-i$	-1	h	g	$-f$	$-e$
e	e	$-f$	g	$-h$	-1	i	$-j$	k
f	f	e	$-h$	$-g$	$-i$	-1	k	j
g	g	$-h$	$-e$	f	j	$-k$	-1	i
h	h	g	f	e	$-k$	$-j$	$-i$	-1

The Octonions



With i the imaginary constant whose square is -1, the set $\{\pm 1, \pm i\}$ forms a *group*: multiplication keeps you inside the set, it allows inverses (e.g. $i \times -i = -i^2 = -(-1) = 1$, so $i^{-1} = -i$) and it is associative (that is, $x \times (y \times z)$ is the same as $(x \times y) \times z$ — the bracketing can safely be forgotten). In 1843, the great Irish scientist William Rowan Hamilton discovered the *quaternions*: i is joined by mysterious companions j and k who multiply according to the circular diagram above left: if x and y follow each other *clockwise* round the circle, then $x \times y = +$ the other quantity; if *anticlockwise*, the product is negative: $ij = k$, $kj = -i$, etc. And, again, $\{\pm 1, \pm i, \pm j, \pm k\}$ is a group. J.T. Graves, a professor of law in London, was inspired to try and go one better: just two months later he had produced the *octonions*, whose multiplication table is given centre and can be constructed from the Fano plane (above right; to keep the diagram simple, only three points from each circle are given: we must imagine $e \to j \to g$, for example, cycling back round to e). But Hamilton spotted a snag: octonion multiplication is not associative. For example, $(ij)e = ke = h$ but $i(je) = i(-g) = -ig = -h$. The octonions were discovered independently by Cayley and are sometimes called Cayley numbers.

A hundred years later, in Germany, Ruth Moufang invented a deep connection between algebra and projective geometry via the idea of a *loop*: exactly those arithmetics which fail to be groups just through being nonassociative. A *Moufang loop* is one in which any x, y and z *nearly* associate: they obey three (equivalent) *Moufang identities*:

$$\text{left: } (xy \cdot x)z = x(y \cdot xz), \qquad \text{middle: } (xy)(zx) = (x \cdot yz)x, \qquad \text{right: } (xy \cdot z)y = x(y \cdot zy).$$

You can check these hold in the octonions which are a classic example of a Moufang loop; the quaternions, hiding associatively inside, are a group thanks to Moufang's theorem. You can check, too, the corollary to Moufang's Theorem, that any Moufang loop is *diassociative*: any *pair* of elements whatsoever generates a group (put $y = 1$ in the left Moufang identity and apply the theorem to x, x and z). E.g. $\{e, f\}$ generate a group of order 8, with elements $\{\pm 1, \pm i, \pm e, \pm f\}$.

Ruth Moufang (1905–1977) played an indirect part in the classification of the finite simple groups: Richard Parker's 1985 construction of a Moufang loop of order 2^{13} was used by John Conway to construct the Monster (order $\approx 8 \times 10^{53}$).

Web link: Stephen M. Gagola III's paper at www.quasigroups.eu/contents/19.php

Further reading: *On Quaternions and Octonions* by J.H. Conway and D.A. Smith, AK Peters, 2003.

Created by Robin Whitty for www.theoremoftheday.org