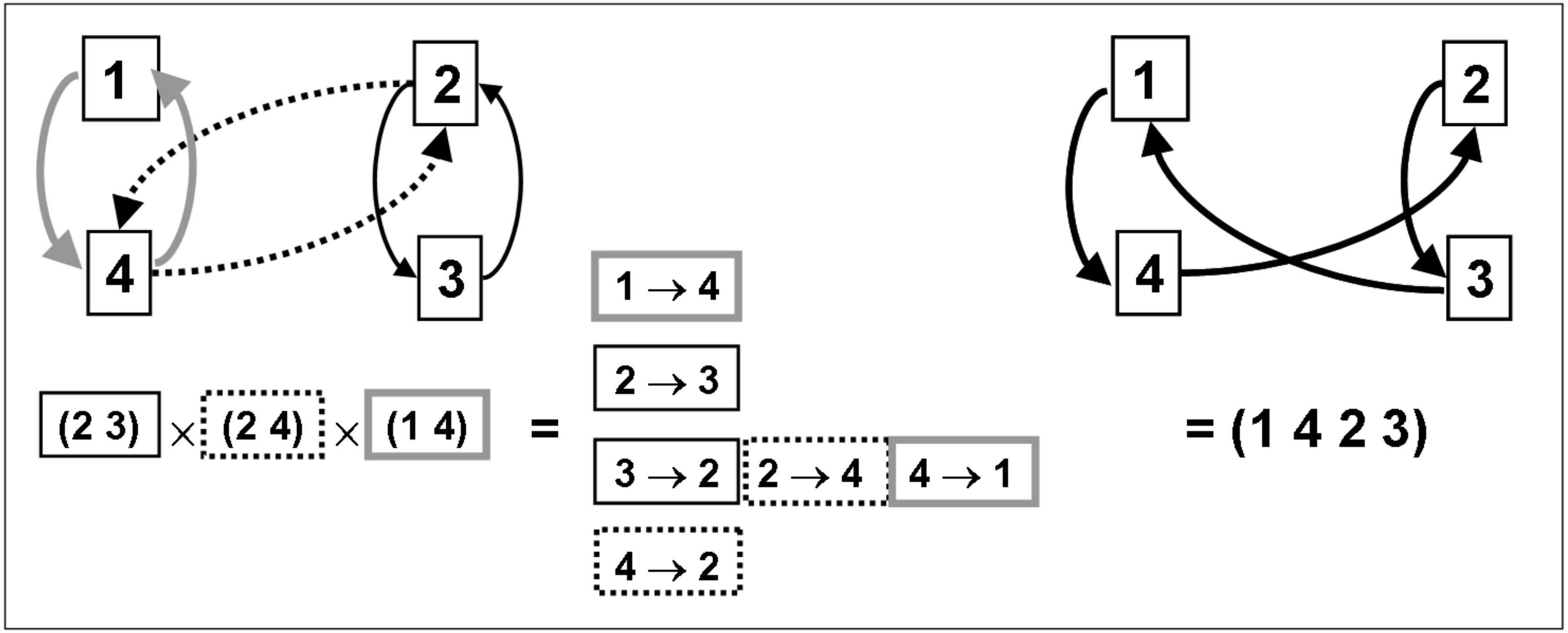
# THEOREM OF THE DAY

**Netto's Conjecture (Dixon's Theorem)** *The proportion of pairs of permutations of n elements which generate the whole symmetric group tends to* $\frac{3}{4}$ *as* $n \to \infty$.




Multiplying permutations is order-dependent. On the left, a diagrammatic depiction is given of the product of three *transpositions* (permutations of two elements): (2 3), (2 4) and (1 4). When distinguished, as in the example here, by different patterns the product permutes an element along the longest path *keeping to the order of the patterns*. So, 1 only gets permuted as far as 4, because this uses a fat grey arrow which is the final one available in the thin black-dotted-fat grey ordering which reflects the multiplication order. But 3 can get all the way to 1.

Any permutation may be written as a product of either an even or an odd number of transpositions but not both. Here, (1 4 2 3) is produced as the product of 3 transpositions and is consequently called an *odd* permutation.

Suppose we take *two* permutations of $n$ elements, $p$ and $q$, at random and build all products by combining $p$ and $q$ as often as we like and in any order we like. How many of the $n!$ permutations constituting the *symmetric group* will we produce all together? If $p$ and $q$ are both even then we will only ever get even permutations. But 3/4 of the time at least one will be odd. Netto conjectured that, when $n$ becomes large, almost all such pairs will generate the whole symmetric group.

Eugen Netto's 1882 conjecture waited nearly a century for a proof. In 1969 Dixon proved that, when $n$ is sufficiently large, the proportion of pairs of permutations of $n$ elements that either generates the symmetric group or generates all possible even permutations (the *alternating group*) exceeds $1 - 2/(\log\log n)^2$. This number approaches 1 very slowly, requiring $n \approx 10^{10^6}$ to give 0.99. Dixon conjectured that the actual proportion would be $1 - 1/n + O(1/n^2)$, reaching 0.99 around $n = 100$. This was proved in 1989 by L. Babai, assuming the classification of the finite simple groups.

**Web link:** mathstat.carleton.ca/ jdixon/Prgrpth.pdf. See www.renyi.hu/~maroti/wiegold.pdf for recent developments regarding probability bounds for the theorem.

**Further reading:** *Introduction to Algebra, 2nd Edition* by Peter J. Cameron, Oxford University Press, 2007, chapter 3.