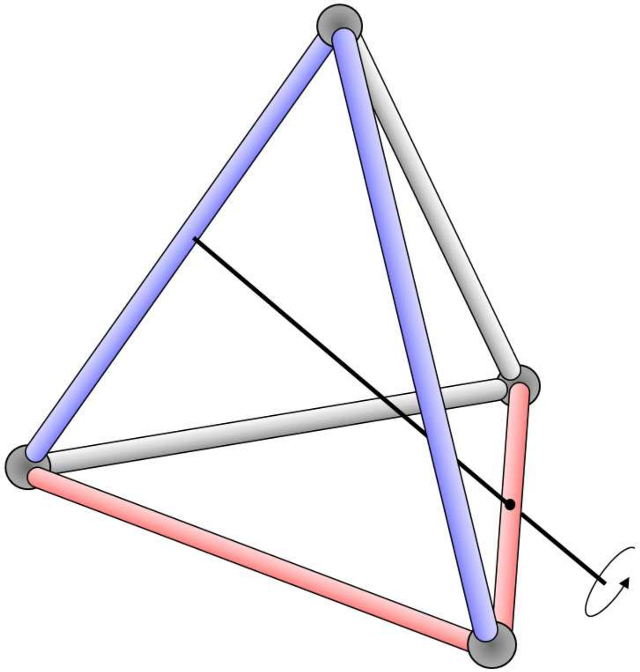
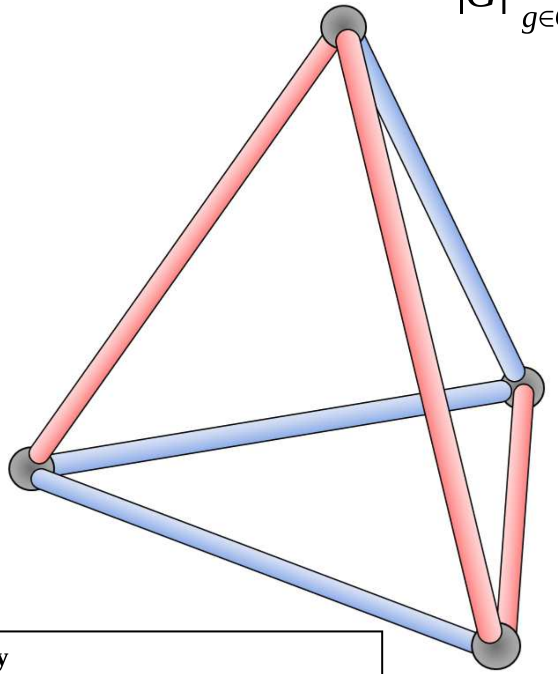
THEOREM OF THE DAY

The Cauchy–Frobenius Lemma *Let G be a finite permutation group on set Ω. For $g \in G$, let $\mathrm{fix}(g)$ be the number of points of Ω not moved by g. Then*

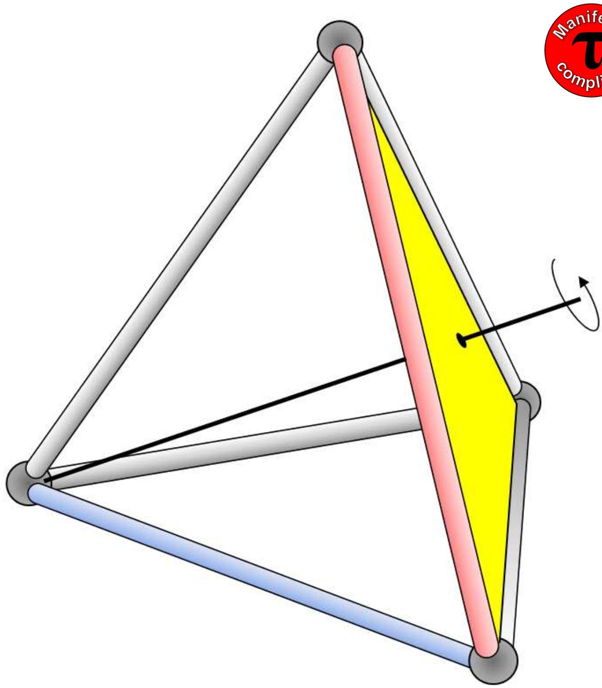
$$\text{number of orbits of } G \text{ on } \Omega = \frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g).$$



Edge-edge rotation ($1/2$-turn, $\tau/2$)
Number of symmetries: 3
Edges unmoved: 2 (both free to colour)
Edges moved: 4 (2 pairs, one free colour each)
Total colourings fixed per symmetry: 2^4



Identity
Number of symmetries: 1
Edges unmoved: 6 (all free to colour)
Edges moved: 0
Total colourings fixed per symmetry: 2^6



Vertex-to-face rotation ($\tau/3, 2\tau/3$)
Number of symmetries: 8
Edges unmoved: 0
Edges moved: 6 (1 free at vertex, 1 free on face)
Total colourings fixed per symmetry: 2^2

A typical visualisation and application of the lemma is shown above. We would like to count red-blue colourings of the edges of a regular tetrahedron, ignoring colourings which are the same under rotational symmetry. For example, the left-hand tetrahedron gets a different colouring from the centre tetrahedron if its uncoloured edges are coloured blue and red with blue at the top vertex. However, with red at the top vertex the result is symmetrical to the centre tetrahedron, by applying to the latter an anticlockwise $\tau/3$ rotation of its right-most face (the rotation illustrated in the right-hand tetrahedron).
Let Ω be the set of 2^6 colourings. The orbits of Ω under the twelve rotational symmetries of the tetrahedron are precisely the different colourings. So we determine $\mathrm{fix}(g)$ for each symmetry g. The results are shown above: for instance, an edge-edge rotation g leaves two edges rotated but in the same location, so any colouring of these edges will be fixed by g. But the other edges are flipped in pairs. For a pair to be fixed both its edges must be the same colour, so each pair only gives one freely colourable edge. Finally the number of orbits is the total number of fixed colourings divided by the order of the group: $\left(3\times 2^4 + 1\times 2^6 + 8\times 2^2\right)/12 = 12$.

The Orbit Counting Lemma is often attributed to William Burnside (1852–1927). His famous 1897 book *Theory of Groups of Finite Order* perhaps marks its first ‘textbook’ appearance but the formula dates back to Cauchy in 1845.

Web link: arxiv.org/abs/2007.15106
Further reading: *Combinatorics* by Peter Cameron, CUP, 1994, chapter 15.