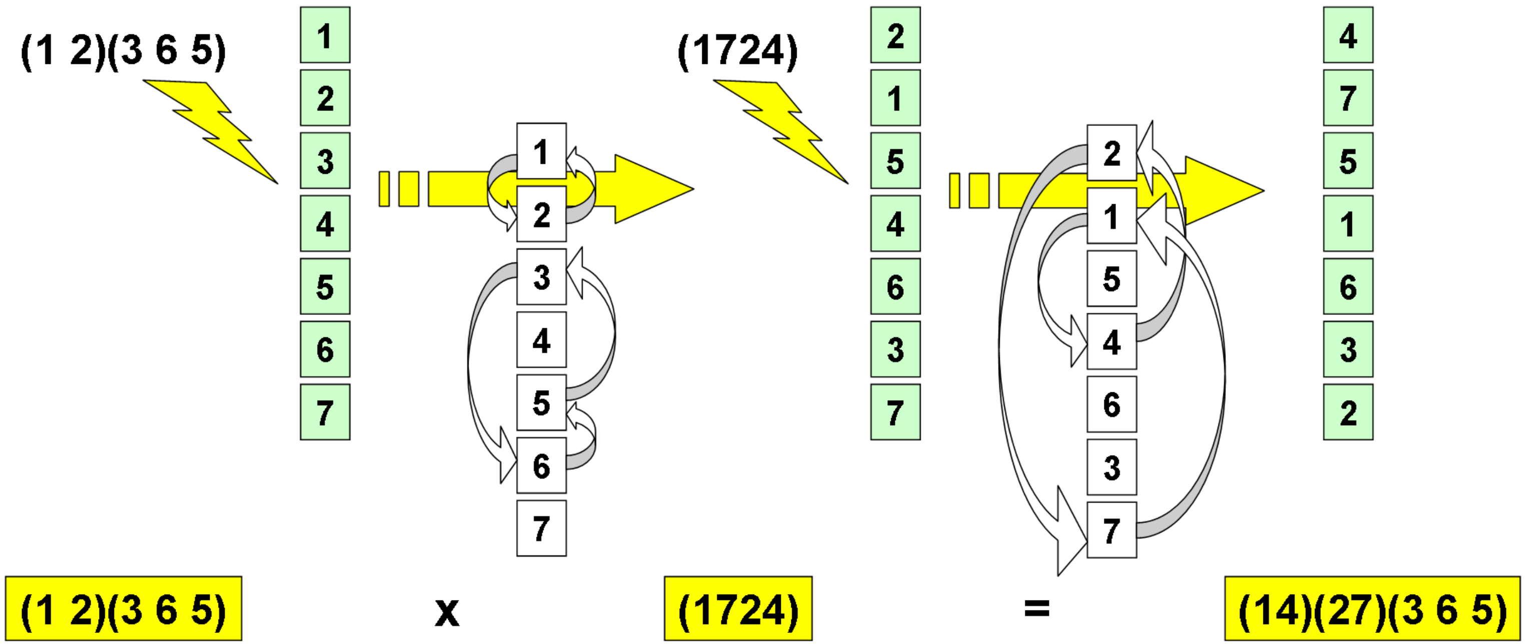
THEOREM OF THE DAY

The Orbit Counting Lemma *Let G be a finite permutation group on set Ω. For $g \in G$, let* fix(g) *be the number of points of Ω not moved by g. Then*

$$\textit{number of orbits of } G \textit{ on } \Omega = \frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g).$$



Here $\Omega = \{1,2,3,4,5,6,7\}$. Permutations on Ω can be thought of as rearrangements of the contents of a line of 7 boxes, one for each element of Ω. A combination, or *product*, of two permutations is shown here; the first has fix((1 2)(3 6 5)) = 2, since only the fourth and seventh boxes' contents are unmoved, while fix((1 7 4 2)) = 3. We get a finite permutation group by starting with a collection of permutations, including the *identity* which moves no box's contents, and successively producing all possible finite products. The two permutations used here generate a group of 24 different permutations, each of which can either move the contents of boxes 3,5 and 6 or the contents of boxes 1, 2, 4 and 7, but without interchanging between these two subsets, which thereby constitute the two orbits of the group. The summation is as follows:

$$\frac{1}{24}(7+5+3+3+3+5+3+3+4+2+0+0+0+2+0+0+4+2+0+0+0+2+0+0) = \frac{1}{24}(48) = 2.$$

The Orbit Counting Lemma is often attributed to William Burnside (1852–1927). His famous 1897 book *Theory of Groups of Finite Order* perhaps marks its first 'textbook' appearance but the formula dates back to Cauchy in 1845.

Web link: undergraduate.csse.uwa.edu.au/units/CITS7209, see Lecture 3 (0.3MB).

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.