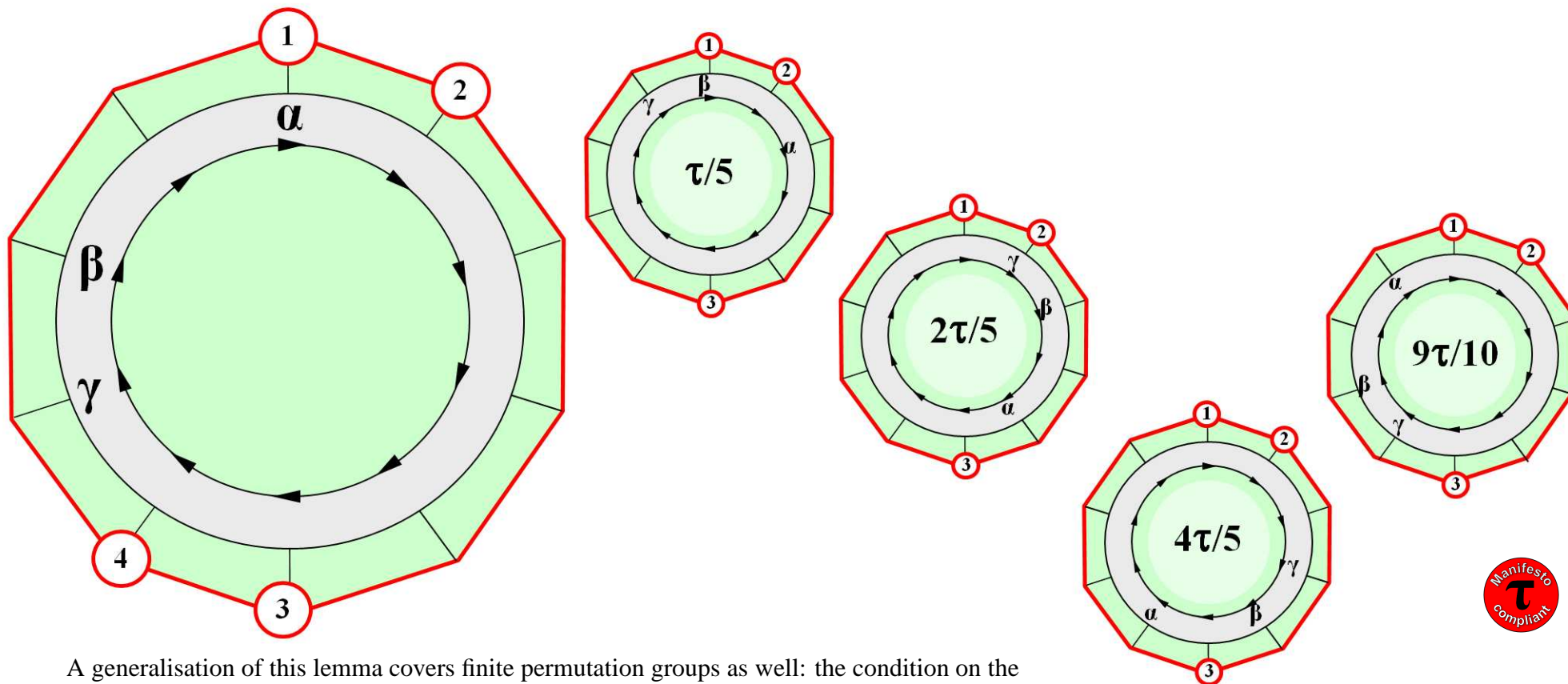




# THEOREM OF THE DAY

**Neumann's Separation Lemma** *Let  $G$  be a permutation group acting on an infinite set  $\Omega$  with no finite orbits. Then for any finite subsets  $\Gamma$  and  $\Delta$  of  $\Omega$  there exists a permutation  $g \in G$  for which  $\Gamma^g \cap \Delta = \emptyset$ .*



A generalisation of this lemma covers finite permutation groups as well: the condition on the orbits is now that no orbit has size less than  $|\Gamma| \times |\Delta|$  (this is automatically satisfied if all orbits are infinite). We have chosen to illustrate this finite version, with the group  $G$  being the rotational symmetries of the 10-point decagon.  $G$  acts *transitively* on  $\Omega$ , the ten vertices of our decagon, that is, every vertex is carried to any other by some rotation. So there is only one orbit and this has size 10. On the left,  $\Gamma = \{\alpha, \beta, \gamma\}$  and  $\Delta = \{1, 2, 3, 4\}$  (the characters have no significance, merely being used to identify our choice of decagon vertices). Since  $|\Gamma||\Delta| = 3 \times 4 = 12 > 10$ , the Separation Lemma is not guaranteed to hold, and indeed it can be checked that no rotation of  $\Gamma$  avoids an overlap with  $\Delta$ . On the right, however, we have reduced  $\Delta$  to  $\{1, 2, 3\}$  and now the Lemma applies, with both  $\Gamma^{3\tau/5} \cap \Delta$  ( $\tau = 2\pi$ ) and  $\Gamma^{9\tau/10} \cap \Delta$  being empty.

Peter M Neumann proved his original lemma in 1974 using a 1954 theorem of his father Bernhard Neumann (1909–2002), also a famous group theorist. The extension to the finite case was done in collaboration with Robert G. Burns, Bryan Birch and Sheila Oates Macdonald in the same year.

**Web link:** [cameroncounts.wordpress.com/lecture-notes/synchronization/](http://cameroncounts.wordpress.com/lecture-notes/synchronization/): a proof of the finite case of the Lemma is given in **Lecture 3**.

**Further reading:** *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.

