THEOREM OF THE DAY

The Second Isomorphism Theorem Suppose $H$ is a subgroup of group $G$ and $K$ is a normal subgroup of $G$. Then $HK$ is a group having $K$ as a normal subgroup, $H \cap K$ is a normal subgroup of $H$, and there is an isomorphism from $H/(H \cap K)$ to $HK/K$ defined by $h(H \cap K) \mapsto hK$.

To illustrate we take $G$ to be $\text{Sym}_5$, the group of $5!$ permutations of $\{1, 2, 3, 4, 5\}$. The Frobenius group $F_{20}$ may be defined as a subgroup $H$ of $G$ generated by a 5-cycle, $a$, and a 4-cycle, $b$, satisfying $(ab)^4 = a(ab)(ba)^{-1} = 1$. We take $K$ to be $\text{Alt}_5$, the subset of $5!/2$ even permutations: identity, 5-cycles, and products of two 2-cycles. This is normal in $G$ (i.e. $g^{-1}Kg = K$ for all $g$) because conjugation, $g^{-1}xg$, is a 1–1 mapping which preserves cycle structure; similarly, $H \cap K$, the subset of even permutations in $H$, is a normal subgroup of $H$.

Now we can uncover the behaviour of the normal subgroup of even permutations of $F_{20}$. The target quotient, $HK/K$, is $\text{Sym}_5/\text{Alt}_5 \cong C_2$ because (1) any coset of $K$, say, $hK$, is either all of $K$ (if $h$ is even) or all odd permutations in $\text{Sym}_5$ (if $h$ is odd), and (2) multiplication of cosets mirrors addition modulo 2: $hK.h'K = hh'K$ switches coset if and only if $h$ and $h'$ have different parity. So cosets of $H \cap K$ must behave in exactly the same way in $H$. And we can see this in the Cayley graph: $H \cap K$ is the ‘identity’ coset consisting of all vertices $(m, n)$, $m$ even; there is one other coset, $b(H \cap K)$, and left multiplication by $b$ cycles between the two.

Web link: people.reed.edu/~jerry/332/09isom.pdf

Further reading: Classic Algebra by P.M. Cohn, John Wiley & Sons, 2000, Chapter 9.

KEY

$a := (1 2 3 4 5)$

$b := (1 2 5 4)$

Vertices (group elements)

$(m, n) := b^m a^n = (1 2 5 4)^m (1 2 3 4 5)^n$

E.g. $(1, 2) = b^1 a^2$

$= (1 2 5 4)(1 2 3 4 5)^2$

$= (1 2 5 4)(1 3 5 2 4)$

$= (1 4 3 5)$

Edges (left multiplication)

- by $a$
- by $b$

(anticlockwise cycle)

This theorem, due in its most general form to Emmy Noether in 1927, is an easy corollary of the first isomorphism theorem. Thus, if $f : H \to HK/K$ is the surjective homomorphism $h \mapsto hK$ then and $H/\ker f \cong \text{im} f$ and $\ker f = H \cap K$.

It is sometimes call the ‘parallelogram rule’ in reference to the diagram on the right.