The Second Isomorphism Theorem

Suppose \( H \) is a subgroup of group \( G \) and \( K \) is a normal subgroup of \( G \). Then \( HK \) is a group having \( K \) as a normal subgroup, \( H \cap K \) is a normal subgroup of \( H \), and there is an isomorphism from \( H/(H \cap K) \) to \( HK/K \) defined by \( h(H \cap K) \leftrightarrow hK \).

To illustrate we take \( G \) to be \( \text{Sym}_5 \), the group of 5! permutations of \( \{1, 2, 3, 4, 5\} \). The Frobenius group \( F_{20} \) may be defined as a subgroup \( H \) of \( G \) generated by a 5-cycle, \( a \), and a 4-cycle, \( b \), satisfying \((ab)^4 = a(ab)(ba)^{-1} = 1\). We take \( K \) to be \( \text{Alt}_5 \), the subset of even permutations: identity, 5-cycles, and products of two 2-cycles. This is normal in \( G \) (i.e. \( g^{-1}Kg = K \) for all \( g \)) because conjugation, \( g^{-1}xg \), is a 1–1 mapping which preserves cycle structure; similarly, \( H \cap K \), the subset of even permutations in \( H \), is a normal subgroup of \( H \).

Now we can uncover the behaviour of the normal subgroup of even permutations of \( F_{20} \). The target quotient, \( HK/K \), is \( \text{Sym}_5/\text{Alt}_5 \cong C_2 \) because (1) any coset of \( K \), say, \( hK \), is either all of \( K \) (if \( h \) is even) or all odd permutations in \( \text{Sym}_5 \) (if \( h \) is odd), and (2) multiplication of cosets mirrors addition modulo 2: \( hK.h'K = hh'K \) switches coset if and only if \( h \) and \( h' \) have different parity. So cosets of \( H \cap K \) must behave in exactly the same way in \( H \). And we can see this in the Cayley graph: \( H \cap K \) is the ‘identity’ coset consisting of all vertices \((m, n)\), \( m \) even; there is one other coset, \( b(H \cap K) \), and left multiplication by \( b \) cycles between the two.

Web link: people.reed.edu/~jerry/332/09isom.pdf
Further reading: Classic Algebra by P.M. Cohn, John Wiley & Sons, 2000, Chapter 9.

This theorem, due in its most general form to Emmy Noether in 1927, is an easy corollary of the first isomorphism theorem. Thus, if \( f : H \to HK/K \) is the surjective homomorphism \( h \mapsto hK \) then and \( H/\ker f \cong \text{im} f \) and \( \ker f = H \cap K \). It is sometimes called the ‘parallelogram rule’ in reference to the diagram on the right.

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