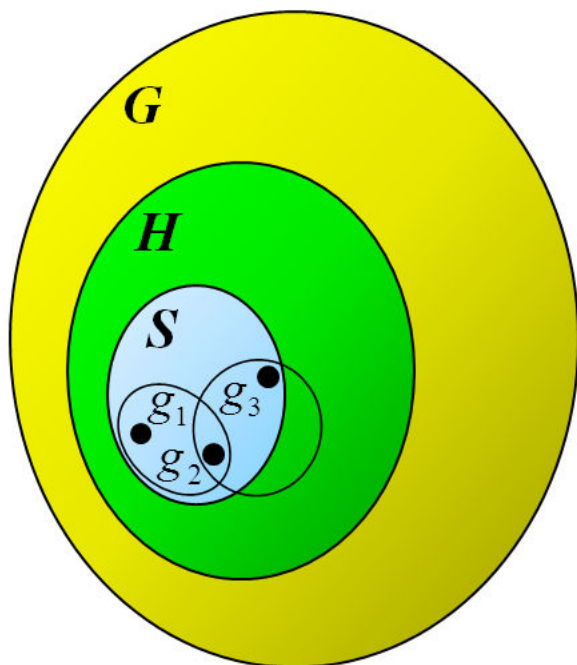




# THEOREM OF THE DAY

**A Theorem on Maximal Sum-Free Sets in Groups** Let  $G$  be a group and let  $H = \langle S \rangle$  be a subgroup generated by a subset  $S$  of the elements of  $G$ . Then unless  $|S| \leq 2$ ,  $S$  cannot be both maximally sum-free and minimal with respect to generating  $H$ .



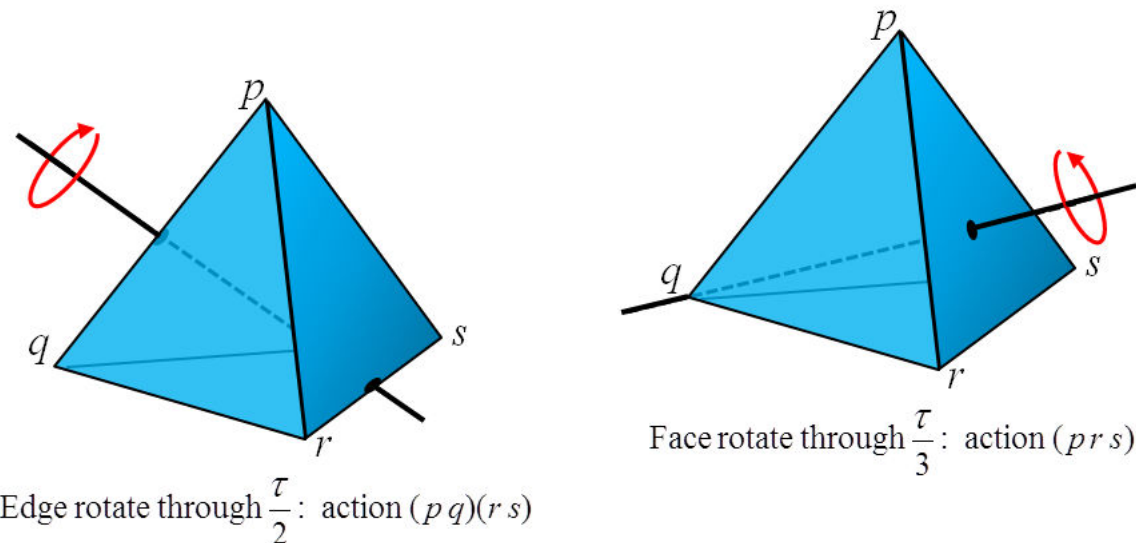
The set-up is illustrated on the left: multiplying two elements in  $S$  may or may not give another element of  $S$ : perhaps  $g_1 g_2 \in S$  but  $g_2 g_3 \notin S$ . If no product  $g_i g_j$  is in  $S$  then  $S$  is called *sum-free* (the terminology derives from the well-established study of  $\mathbb{Z}$ , as an abelian group under addition).

The rotational symmetries

of the regular tetrahedron provide an example, above. There are three rotations though one half-turn ( $\tau/2$  radians) about the axes joining opposite edges; these form a subgroup of four symmetries (including the identity, 1, the null rotation): the result of any sequence of them is again such a rotation. The remaining eight rotations, illustrated far right,

are rotations of  $\pm\tau/3$  about the axes through a face and opposite vertex. This subset does not form a subgroup since it is not closed. For example, rotate the tetrahedron clockwise about face  $pqr$ : the axis in the picture is now pointing vertically down; a clockwise rotation about this axis (looking downwards) gives an identical final result to the  $\tau/2$  opposite-edge rotation switching  $p$  with  $q$  and  $r$  with  $s$ . In terms of the action of the symmetries on vertices we may write this as  $(p r s) \times (p r q) = (p q)(r s)$ . The resulting symmetry does not belong to the set of face-vertex rotations.

Take  $S$  in the theorem to be  $\{(p r s), (p q)(r s), (p s)(q r)\}$ . We can calculate the multiplication table for  $S$ , as shown on the far right. The table reveals  $S$  to be a sum-free set: no product of any two elements is again in  $S$ . And it can be checked that  $S$  is maximal with respect to this property (already the entries in the table exclude seven permutations from extending  $S$ ). The theorem then concludes that whatever is the subgroup of symmetries generated by  $S$ , it may be generated by some two elements of  $S$ . And indeed  $\langle S \rangle$  consists of the whole group of 12 rotational symmetries, and this whole group may be generated by the set  $\{(p r s), (p q)(r s)\}$ .



Edge rotate through  $\frac{\tau}{2}$ : action  $(p q)(r s)$

Face rotate through  $\frac{\tau}{3}$ : action  $(p r s)$

$\times$	$(p r s)$	$(p q)(r s)$	$(p s)(q r)$
$(p r s)$	$(p s r)$	$(p s q)$	$(p q r)$
$(p q)(r s)$	$(p q r)$	1	$(p r)(q s)$
$(p s)(q r)$	$(q s r)$	$(p r)(q s)$	1

Although sum-free sets in the integers have been intensively studied since the 1970s, the extension of this work to general groups is much more recent. In this 2006 study by Michael Giudici and Sarah Hart, the sum-free property reveals an unexpected trade-off of maximality against minimality with respect to a different multiplicative property. The exceptional cases with  $|S| \leq 2$  were identified by Giudici and Hart: there are only eleven groups involved, none having order exceeding 16.

**Web link:** [www.combinatorics.org/ojs/index.php/eljc/article/view/v16i1r59](http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i1r59)

**Further reading:** *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.

