Bondy’s Subset Theorem  Let $S$ be a set with $n$ elements and suppose that $n$ distinct subsets of $S$ are chosen. Then there is a restriction to $n - 1$ elements of $S$ under which these subsets remain distinct.

For $n = 3$ we can state the theorem as: if three distinct vertices are chosen on the cube then some projection on to one of its faces will retain three distinct vertices. In the example below the projection on to the front face will reduce the number of vertices to 2; but the projection on to the top face will retain three vertices.

This geometric version of Bondy’s theorem is valid in general and is further illustrated above for $n = 4$ by applying his elegant proof method to a set of vertices on the 4-cube. In fact, we will try it with $n + 2 = 6$ vertices, highlighted with red squares. Consider all 4-cube edges joining these vertices: each such edge consists of a move along one of the four axes, $x, y, z$ and $w$. Make a list of the axes involved. Starting bottom right and moving clockwise we have: $w, z, x, w, x$. Because there is a cycle, every axis must appear an even number of times in order for us to return to our starting point. So we have used $6/2 < 4$ axes. But this means we can project on the missing axis, because any two vertices identified by such a projection would necessarily have been joined by a 4-cube edge.

And indeed, ‘shrinking’ all $y$ edges in the diagram above retains six distinct vertices. However, this approach is only guaranteed to work with $n$ vertices because in this case either there is a cycle, or the edges joining them will form a forest which, on $n$ vertices, will have at most $n - 1$ edges, guaranteeing the spare axis for our projection.

It is easy, in the above diagram, to find a path on five vertices, with four edges each taking a different axis, so that any projection will identify two vertices.

Bondy’s 1972 theorem answered a question of András Hajnal, part of a wider investigation into extremal properties of set restrictions. Another notable example is: given a collection $F$ of distinct vertices on the $n$-cube, what is the largest $d$ such that some projection on $n - d$ dimensions results in a $d$-cube. The answer, call it $d(F)$, is a measure of the ‘density’ of $F$, called the Vapnik–Chervonenkis-dimension, an important concept in the theory of machine learning. For example, if $F$ is the 6-cycle depicted above then $d(F) = 2$: if we project on the $y$ and $w$ axes we get a 2-cube (a square). A well-known bound is given by Sauer’s Lemma: $|F| \leq \sum_{i=0}^{d(F)} \binom{n}{i}$. For our collection $F$, Sauer’s Lemma says that $|F| \leq 1 + 4 + 6 = 11$, and here we can in fact adjoin another five vertices of the 4-cube before we necessarily have a projection onto a 3-cube.

Web link:  
Perso.limos.fr/ffoucaud/Talks/index.html: see, e.g. the invited talk “Graph identification problems” (2MB pdf).

Further reading:  