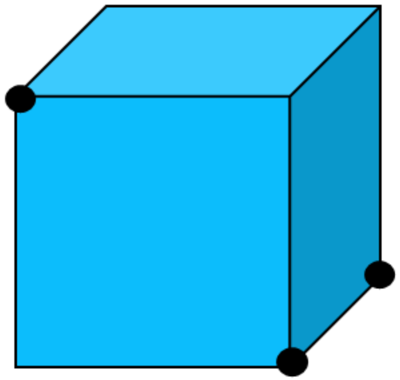
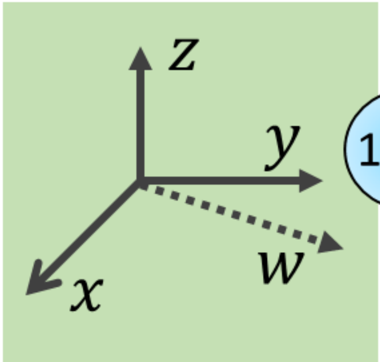
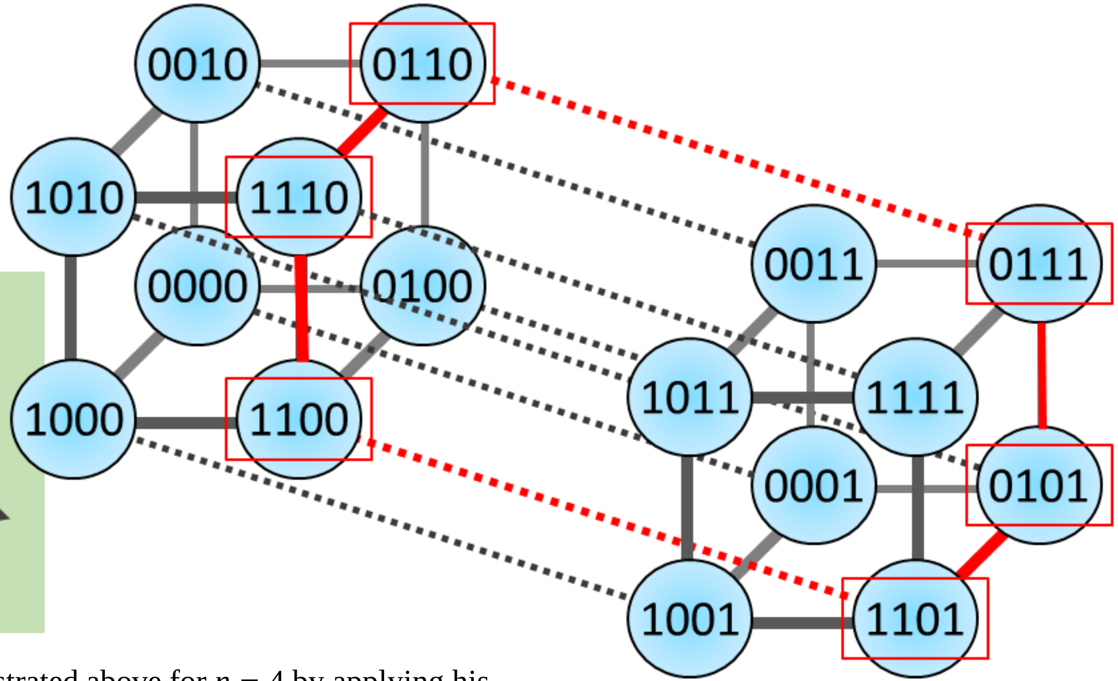
THEOREM OF THE DAY

Bondy's Subset Theorem *Let S be a set with n elements and suppose that n distinct subsets of S are chosen. Then there is a restriction to $n-1$ elements of S under which these subsets remain distinct.*

For $n = 3$ we can state the theorem as: *if three distinct vertices are chosen on the cube then some projection on to one of its faces will retain three distinct vertices*. In the example below the projection on to the front face will reduce the number of vertices to 2; but the projection on to the top face will retain three vertices.







This geometric version of Bondy's theorem is valid in general and is further illustrated above for $n = 4$ by applying his elegant proof method to a set of vertices on the 4-cube. In fact, we will try it with $n + 2 = 6$ vertices, highlighted with red squares. Consider all 4-cube edges joining these vertices: each such edge consists of a move along one of the four axes, x, y, z and w. Make a list of the axes involved. Starting bottom right and moving clockwise we have: $w\,z\,x\,w\,z\,x$. Because there is a cycle, every axis must appear an even number of times in order for us to return to our starting point. So we have used $6/2 < 4$ axes. But this means we can project on the missing axis, because any two vertices identified by such a projection would necessarily have been joined by a 4-cube edge. And indeed, 'shrinking' all y edges in the diagram above retains six distinct vertices. However, this approach is only guaranteed to work with n vertices because in this case either there is a cycle, or the edges joining them will form a forest which, on n vertices, will have at most $n-1$ edges, guaranteeing the spare axis for our projection. It is easy, in the above diagram, to find a path on five vertices, with four edges each taking a different axis, so that any projection will identify two vertices.

Bondy's 1972 theorem answered a question of András Hajnal, part of a wider investigation into extremal properties of set restrictions. Another notable example is: given a collection F of distinct vertices on the n-cube, what is the largest d such that some projection on $n - d$ dimensions results in a d-cube. The answer, call it $d(F)$, is a measure of the 'density' of F, called the Vapnik–Chervonenkis-dimension, an important concept in the theory of machine learning. For example, if F is the 6-cycle depicted above then $d(F) = 2$: if we project on the y and w axes we get a 2-cube (a square). A well-known bound is given by **Sauer's Lemma:** $|F| \leq \sum_{i=0}^{d(F)} \binom{n}{i}$. For our collection F, Sauer's Lemma says that $|F| \leq 1 + 4 + 6 = 11$, and here we can in fact adjoin another five vertices of the 4-cube before we necessarily have a projection onto a 3-cube.

Web link: perso.limos.fr/ffoucaud/Talks/index.html: see, e.g. the invited talk "Graph identification problems" (2MB pdf).

Further reading: *Extremal Combinatorics: With Applications in Computer Science* by Stasys Jukna, 2nd edition, Springer, 2011.