THEOREM OF THE DAY

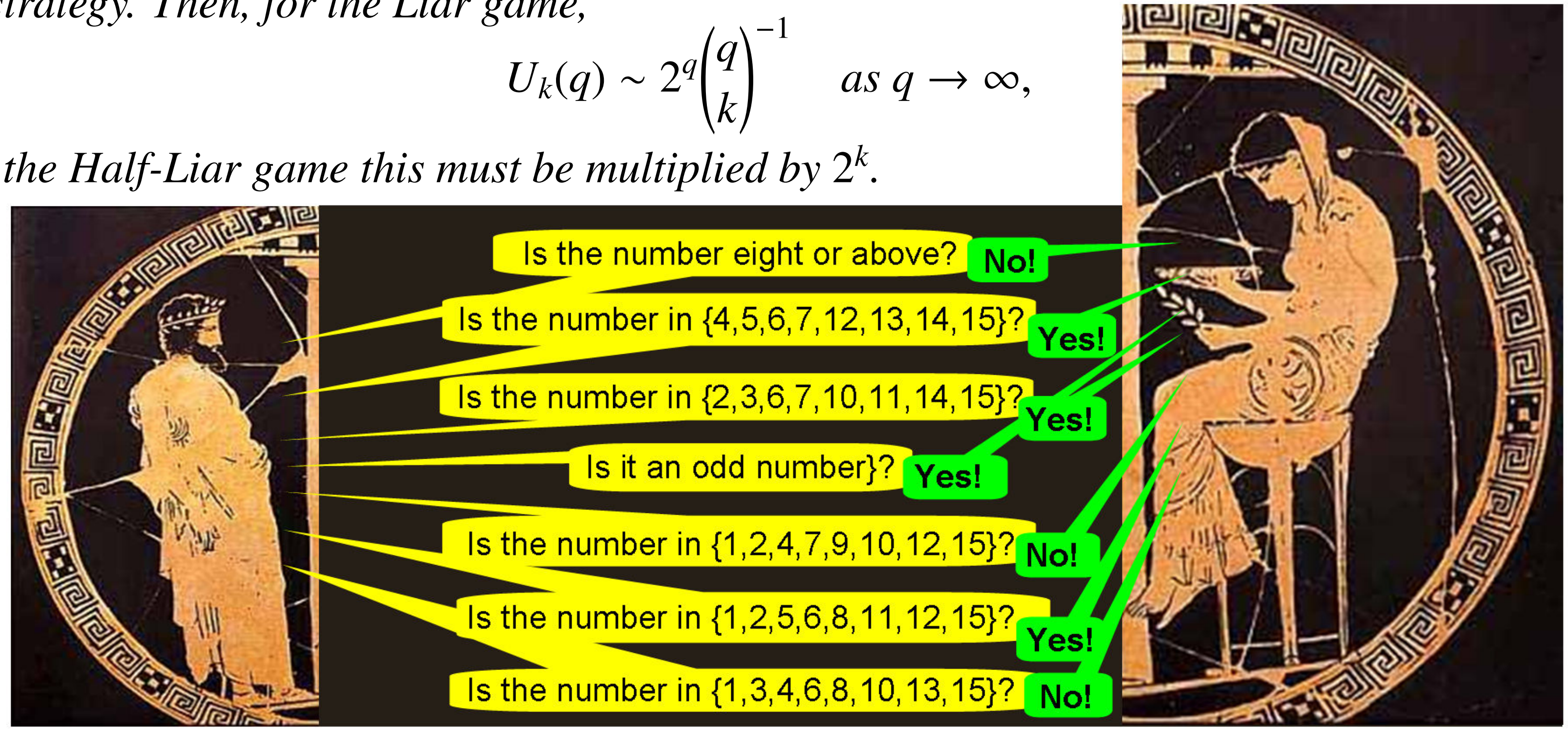
The Asymptotic (Half) Liar Formula *Carole and Paul play the following 'Liar' game: Carole picks a value $x \in \{0, \ldots, n\}$; Paul wins if he identifies x by asking no more than q Yes-No questions of the form "Is x in subset X?"; Carole may lie at most k times, in trying to prevent this. In the 'Half-Liar' game, her lies may only be Yes's. Denote by $U_k(q)$ the maximum value of n for which Paul has a guaranteed winning strategy. Then, for the Liar game,*

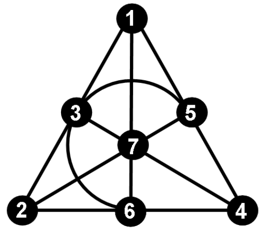
$$U_k(q) \sim 2^q \binom{q}{k}^{-1} \quad \text{as } q \to \infty,$$

while for the Half-Liar game this must be multiplied by 2^k.

Half the world's mischief,
And folly and woe,
Comes from a Yes
That should be a No!

Traditional





Lines:

1 2 3   1 6 7   1 4 5
2 4 6   2 5 7   3 4 7
3 5 6

If Paul can ask the $q = 7$ Y/N questions shown on the right, and provided Carole can lie at most $k = 1$ times, then he can identify her x value for $n = 15$ (so that $U_1(7) \geq 15$). A linear error-correcting code based on the Fano plane (far right) will identify any lie and the correct value of x. Firstly, how to detect a lie: (1) note which questions get a Y answer. If there are three or fewer Ys proceed as follows; otherwise interchange the roles of Y and N in what follows. (2) If there are zero Ys, or three Ys whose positions form a line in the Fano plane, there are no lies; one Y must always be a lie; for two Y positions there will be a unique third position forming a Fano plane line and this is a lie; finally, for three Ys *not* on a line, use the four N positions: three will form a line and the fourth is the lie position. Once Carole's lie, if she made one, is corrected, we may write the correct sequence of Ys and Ns as a binary string of length 7 (Y=1, N=0). Take the first four positions: they encode a binary number between 0 (i.e., 0000) and 15 (i.e., 1111). This is x.

The Liar Game was first divised by Rényi in 1961 and popularised by Ulam in the 70s. Joel Spencer gave the complete asymptotic solution in 1992 and, with Ioana Dumitriu, found the elegant extension to the Half Liar version 10 years later.

Web link: www.math.tamu.edu/~catherine.yan/Files/Halflie.pdf. The perfect strategy for $n = 15$, $q = 7$ and $k = 1$, as illustrated here, is from www.maths.qmul.ac.uk/~pjc/slides/lmspop.pdf. The Delphic Oracle image is from the Staatliche Museen Berlin. The solution to the puzzle posed above may be found here.

Further reading: *Enumerative Combinatorics, Vol. 2,* by R.P. Stanley, Cambridge University Press, 2001.