The Cantor–Bernstein–Schröder Theorem
Let A and B be sets for which there exist injective mappings from A to B and from B to A. Then there is a bijective correspondence between A and B.

We have chosen here a very simple example but one which allows us to follow through the proof of the theorem. Our sets A and B are the real numbers \( \mathbb{R} \), with A represented by the horizontal, \( x \)-axis and B by the vertical, \( y \)-axis. Our injections are \( f : A \rightarrow B \) and \( g : B \rightarrow A \) defined by \( f(x) = e^{-x} \) and \( g(y) = y \), respectively. Of course, \( g \) is already a bijection between A and B: it matches every point on the \( y \)-axis with the exactly corresponding point on the \( x \)-axis. But \( f \) is not: it maps the \( x \)-axis to the positive \( y \)-axis, as indicated by the green arrows at \( \mathbb{R} \). The proof extends \( f \) to a bijection by combining it with \( g \).

The idea is to define a function \( F \) on subsets of A thus:

\[
F(X) = A \setminus g(B \setminus f(X)).
\]

From \( \mathbb{R} \) onwards this is iterated: by \( 5 \) we have constructed \( F(A) \) and are in the process of constructing \( F^2(A) \). The crux is, it can be shown that \( A_0 = A \cap F(A) \cap F^2(A) \cap \ldots \) is a fixed point of \( F \), i.e. \( F(A_0) = A_0 \). This means that \( A \setminus g(B \setminus f(A_0)) = A_0 \) so \( g(B \setminus f(A_0)) = A \setminus A_0 \). Et voilà,

\[
f_0(x) = \begin{cases} f(x) & \text{if } x \in A_0 \\ g^{-1}(x) & \text{if } x \in A \setminus A_0 \end{cases}
\]

is a well-defined bijection from A to B. In our example \( A_0 \) is a single point: the solution to the equation \( e^{-x} = x \), the so-called Omega constant, \( \Omega \approx 0.57 \). Result: in our example, \( f_0 \) and \( g \) are identical!