**THEOREM OF THE DAY**

**Cantor’s Theorem** The power set $2^X$ of a set $X$ cannot be put into one to one correspondence with $X$. Thus the cardinality of $2^X$ is strictly greater than that of $X$.

**Proof:** Think of the elements of $X$ as some, possibly infinite, collection of people. The power set $2^X$ is the set of all subsets of $X$ and we can think of these as all possible communities made up from these people. Now imagine putting the people into one to one correspondence with these possible communities — that is, each person is assigned a unique community and vice versa. If a person is assigned to a community to which they happen to belong then call them a *guard*, otherwise call them a *spy*. The community consisting of all spies is itself a (possibly infinite) community. Is it assigned to a guard or a spy? Neither! A spy would belong to the community, so would be a guard; a guard would *not* belong, so would be a spy. This contradiction proves that the one to one correspondence cannot exist. QED.

This theorem about different ‘sizes’ of infinity strengthens Cantor’s Uncountability Theorem which asserts that the power set of a *countably infinite* set is uncountable. The above argument is essentially another manifestation of the *diagonalisation method*: assume some kind of listing; produce a new object for the list from existing listed objects; show that the new object invalidates the listing. The result is a well-known mathematical phenomenon: an easy proof of a deep and conceptually difficult theorem. The notation $\mathcal{P}(X)$ is often used for power set; the notation $2^X$ is suggested by the fact that, for a finite set of size $n$, the set of all (finite) subsets has size $2^n$. This follows using an ‘include/exclude’-type argument that we see extended to the infinite case in the Uncountability Theorem.

**Web link:** [www.math.hawaii.edu/~dale/godel/godel.html](http://www.math.hawaii.edu/~dale/godel/godel.html)


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