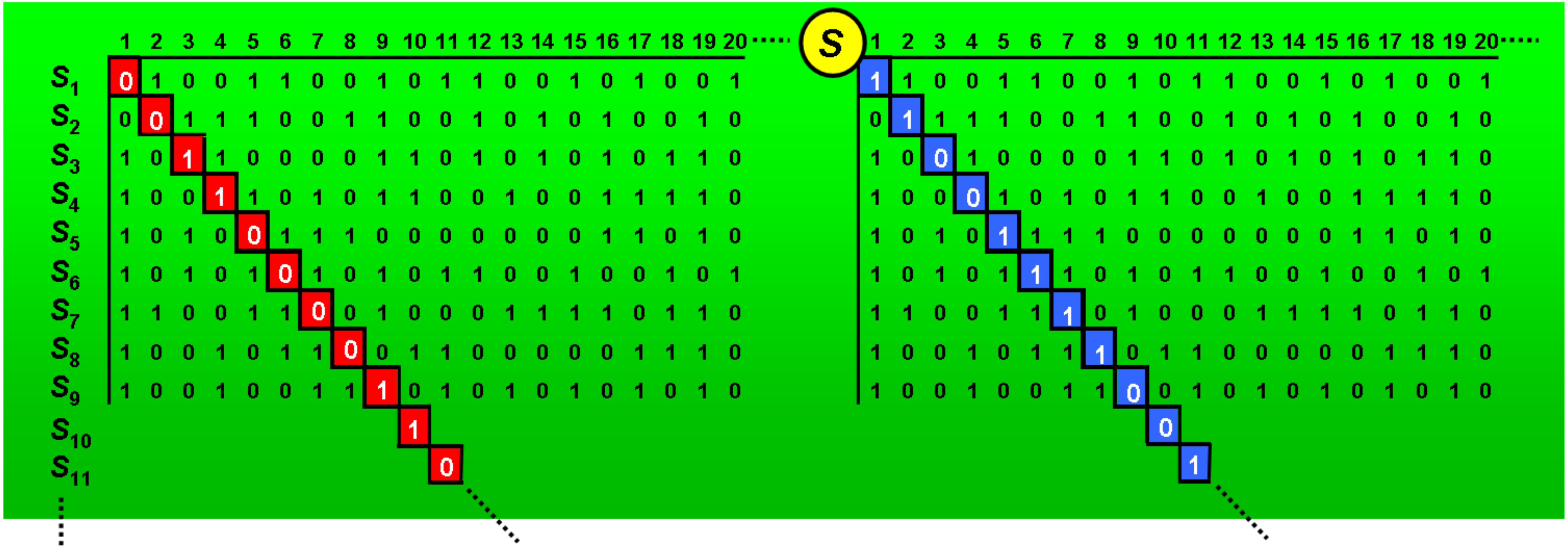
Cantor's Uncountability Theorem *There are uncountably many infinite 0-1 sequences.*



Proof: Suppose you *could* count the sequences. Label them in order: $S_1, S_2, S_3, \ldots,$ and denote by $S_i(j)$ the j-th entry of sequence S_i. Now define a new sequence, S, whose i-th entry is 1, if $S_i(i) = 0$ and 0, if $S_i(i) = 1$. More succinctly, for $i = 1, 2, 3, \ldots,$ $S(i) = 1 - S_i(i)$. Now S is certainly an infinite sequence of 0s and 1s. So it must appear in our list: it is, say, S_k. But then its k-th entry is $S_k(k)$ and this is, by definition, $1 - S_k(k) \neq S_k(k)$. Faced with this contradiction we must conclude that our enumeration of sequences is not, after all possible. QED.

We can see Cantor's famous 'diagonal method' in action in the above illustration. The sequence S generated on the diagonal on the right looks at first like S_7 on the left: "110011 . . .". But the comparison fails at the 7th element, as indeed it must, by definition of S.

The theorem establishes that the set of real numbers is *uncountable* — that is, the real numbers cannot be enumerated in a list indexed by the positive integers (1, 2, 3, . . .). To see this informally, consider the infinite sequences of 0s and 1s to be the binary expansions of fractions (e.g. $0.010011\ldots = 0/2 + 1/4 + 0/8 + 0/16 + 1/32 + 1/64 + \ldots$). More generally, it says that the set of subsets of a countably infinite set is uncountable, and to see *that*, imagine every 0-1 sequence being a different recipe for building a subset: the i-th entry tells you whether to include the i-th element (1) or exclude it (0).

1873 saw Georg Cantor's landmark discovery that there are different 'orders' of infinity: the uncountable set of real numbers contrasting with the sets of rational numbers and algebraic numbers which he showed are both countable.

Web link: www.math.hawaii.edu/~dale/godel/godel.html.

Further reading: *Mathematics: the Loss of Certainty* by Morris Kline, Oxford University Press, New York, 1980.